Solid Mechanics - 202041

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Strength of Materials:

Unit III

Presented by

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Unit III Stresses, Slope & Deflection on Beams [12 Hr.]

Bending Stress on a Beam: Introduction to bending stress on a beam with application, Theory of Simple bending, assumptions in pure bending, derivation of flexural formula, Moment of inertia of common cross section (Circular, Hollow circular, Rectangular, I & T), Bending stress distribution along the same cross-section

Shear Stress on a Beam: Introduction to transverse shear stress on a beam with application, shear stress distribution diagram along the Circular, Hollow circular, Rectangular, I & T cross-section

Slope & Deflection on a Beam: Introduction to slope & deflection on a beam with application,

Session I: Stresses in Machine Elements

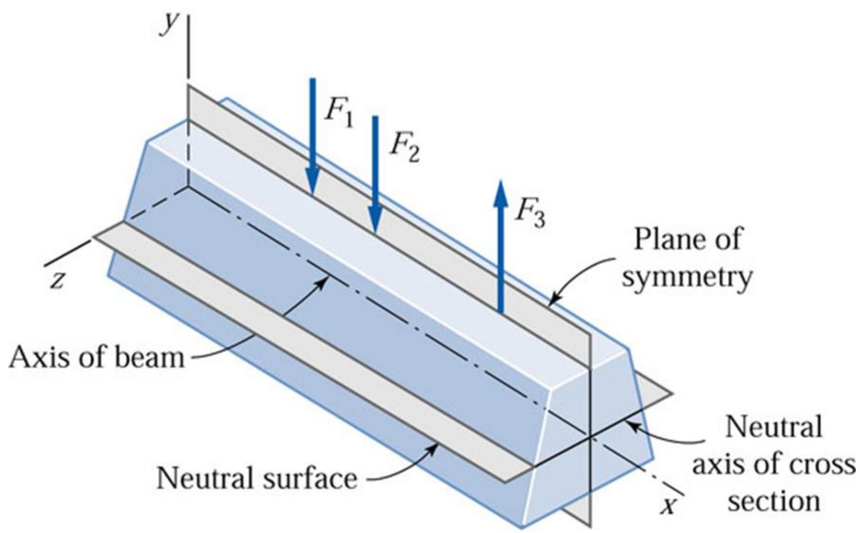
Bending Stresses:

- Theory of simple bending with assumptions
- Derivation of flexural formula
- Second moment of area of common cross sections (rectangular, I,T,C) with respect to centroidal and parallel axes
- Bending stress distribution diagrams
- Moment of resistance and section modulus

Theory of simple bending

The conditions for using theory of simple bending are:

1. The beam is subjected to pure bending
2. Shear force is zero
3. No torsional or axial loads are present
4. Material is isotropic (or orthotropic) and homogeneous



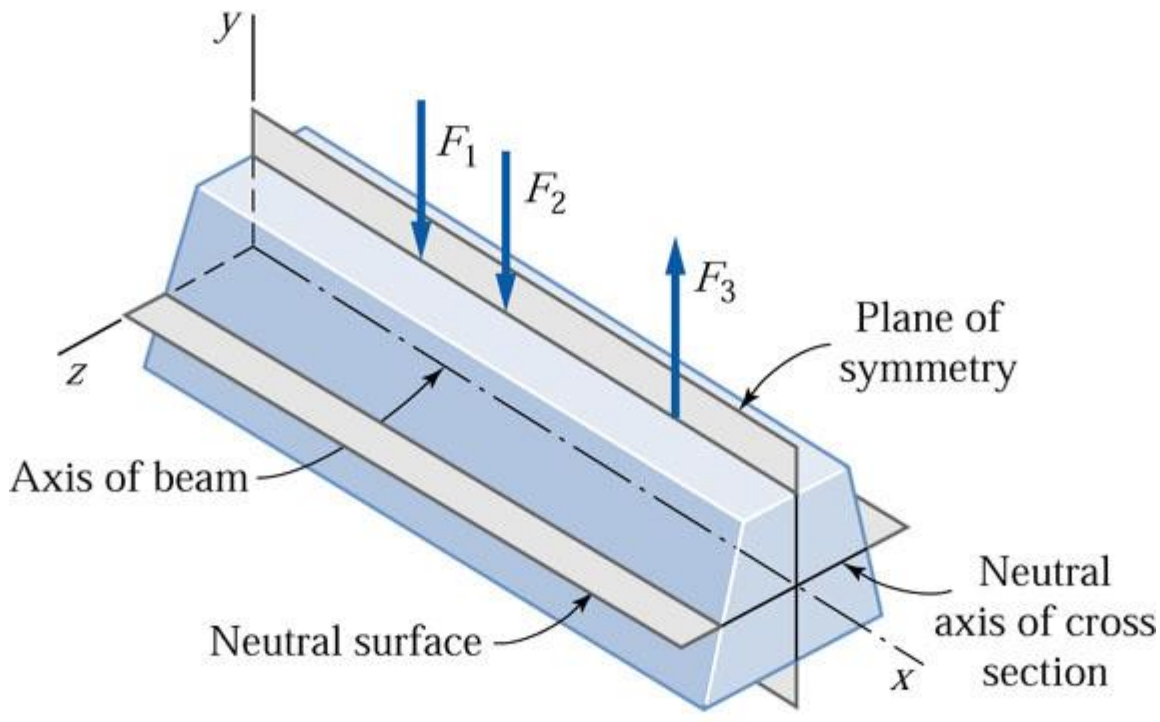
Assumptions

- Beam is initially straight and has a const. cross-section
- Beam is made up of homogeneous material (isotropic)
- Beam has a longitudinal plane of symmetry
- Geometry of the beam is such that bendingis the primary cause of failure

not buckling Elastic limit is nowhere exceeded

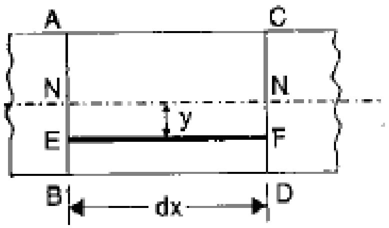
- E is same in tension and compression
- Plane cross-section remains plane before and after bending
- The radius of curvature is large compared with the dimensions of the cross-section.

Beam Contd.

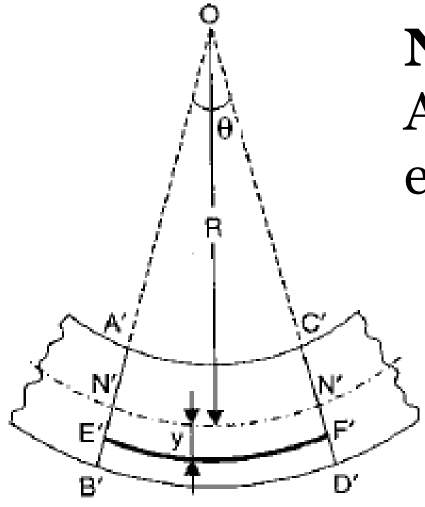


y
z
F_1
F_2
F_3
Plane of symmetry
Axis of beam
Neutral surface
Neutral axis of cross section
x

Theory of Simple Bending



(a)



(b)

Neutral layer or surface (N-N):
A layer which is neither shortened nor elongated

Neutral axis (N-A):
line of intersection of neutral layer on a cross-section of beam is known as neutral axis

Below NN: Tension, **Above NN:** Compression

The amount by which a layer increases or decreases in length, depends upon the position of the layer w.r.t. N-N. This theory of bending is known as **theory of simple bending**.

R = Radius of neutral layer N'-N'.
θ = Angle subjected at O by A'B' and C'D' produced.
y = Distance from the neutral layer.

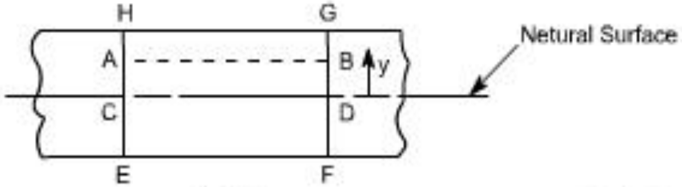


Fig 1(a)

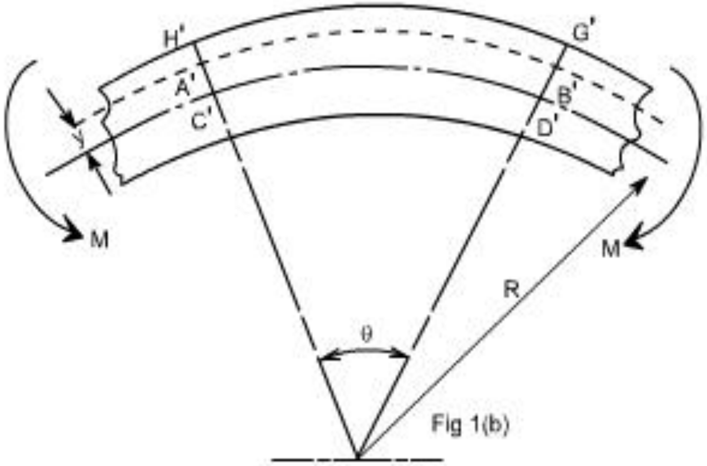


Fig 1(b)

Therefore,

$$\text{strain in fibre AB} = \frac{\text{change in length}}{\text{orginal length}}$$

$$= \frac{A'B' - AB}{AB}$$

But AB = CD and CD = C'D'

refer to fig1(a) and fig1(b)

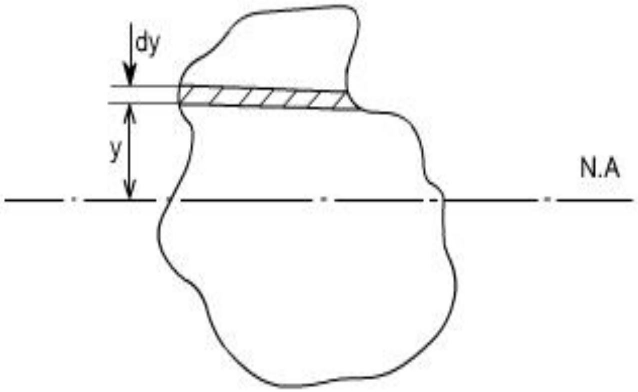
$$\therefore \text{strain} = \frac{A'B' - C'D'}{C'D'}$$

$$= \frac{(R+y)\theta - R\theta}{R\theta} = \frac{R\theta + y\theta - R\theta}{R\theta} = \frac{y}{R}$$

However $\frac{\text{stress}}{\text{strain}} = E$ where E = Young's Modulus of elasticity

Therefore, equating the two strains as obtained from the two relations i.e,

$$\frac{\sigma}{E} = \frac{y}{R} \text{ or } \frac{\sigma}{y} = \frac{E}{R} \qquad \ldots\ldots\ldots\ldots(1)$$



$\sigma = \frac{E}{R} y$

if the shaded strip is of area 'dA'
then the force on the strip is

$F = \sigma \, \delta A = \frac{E}{R} y \, \delta A$

Moment about the neutral axis would be $= F.y = \frac{E}{R} y^2 \delta A$

The toatl moment for the whole cross-section is therefore equal to

$M = \sum \frac{E}{R} y^2 \delta A = \frac{E}{R} \sum y^2 \delta A$

$$M = \frac{E}{R} I \qquad \ldots\ldots\ldots(2)$$

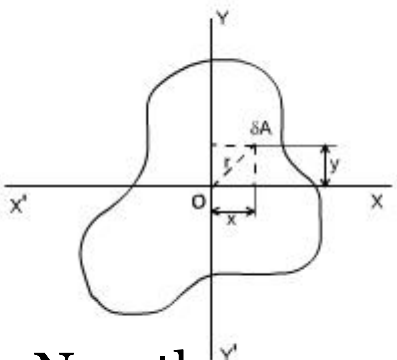
combining equation 1 and 2 we get

$$\boxed{\frac{\sigma}{y} = \frac{M}{T} = \frac{E}{R}}$$

Second Moment of Area

- Taking an analogy from the mass moment of inertia, the second moment of area is defined as the summation of areas times the distance squared from a fixed axis
- This property arised while we were driving bending theory equation
- This is also known as the moment of inertia
- An alternative name given to this is second moment of area, because the first moment being the sum of areas times their distance from a given axis and the second moment being the square of the distance
- Term: $\int y^2 \, dA$ is called second moment of area

Second Moment of Area contd.



Consider any cross-section having small element of area dA.

Then By Definition,

I_x(Mass Moment of Inertia about x-axis = $\int y^2\, dA$

I_y(Mass Moment of Inertia about y-axis = $\int x^2\, dA$

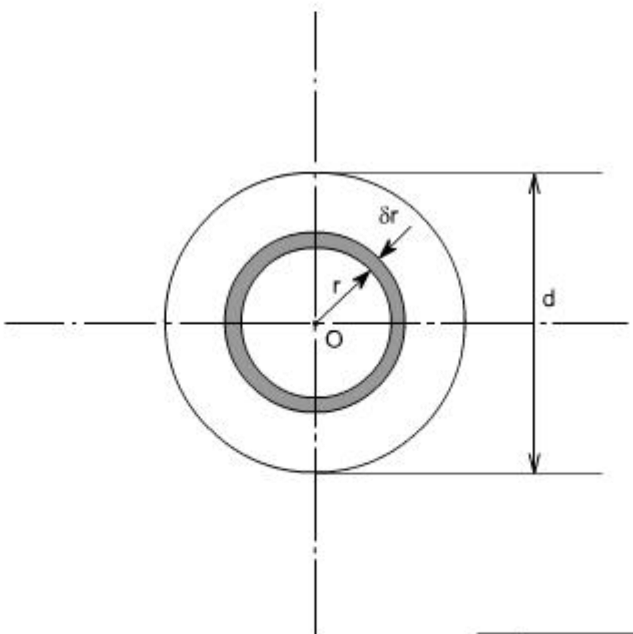
Now the moment of inertia about an axis through 'O' and perpendicular to the plane of figure is called the polar moment of inertia (J).
(The polar moment of inertia is also the area moment of inertia).
The polar moment of inertia is given by

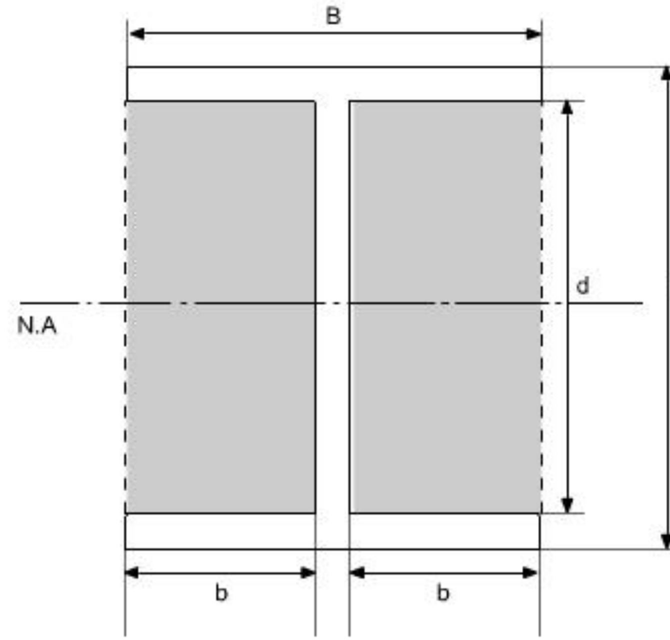
$$= \int r^2 dA$$
$$= \int (x^2 + y^2) dA$$
$$= \int x^2 dA + \int y^2 dA$$
$$= I_X + I_Y$$
$$\text{or } J = I_X + I_Y \qquad \ldots\ldots\ldots (1)$$

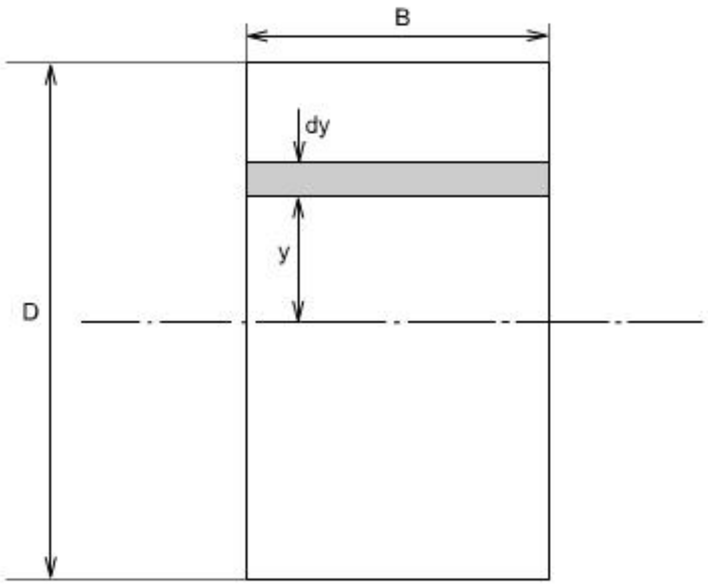
The relation (1) is known as the **perpendicular axis theorem** and may be stated as follows:

The sum of the Moment of Inertia about any two axes in the plane is equal to the moment of inertia about an axis perpendicular to the plane, the three axes being concurrent, i.e, the three axes exist together.

Second Moment of Area contd.







Section Modulus and Moment of Resistance

It is the ratio of moment of inertia of a section about the neutral axis to the distance of the outermost layer from the neutral axis.

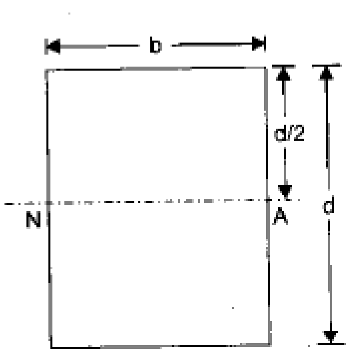
$$Z = \frac{I}{y_{max}}$$

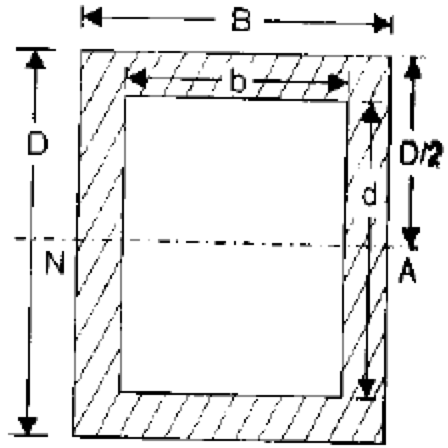
I = M.O.I. about neutral axis
y_{max} = Distance of the outermost layer from the neutral axis

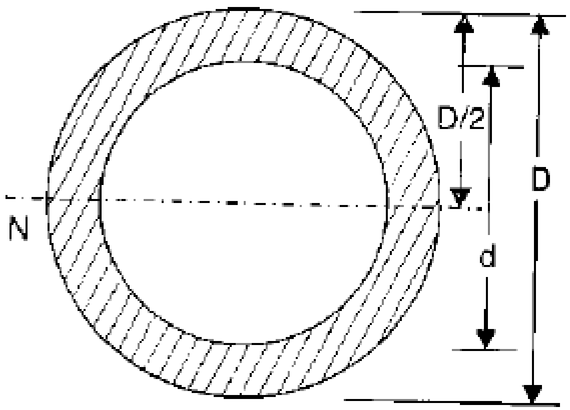
$$\therefore \quad M = \sigma_{max} . Z$$

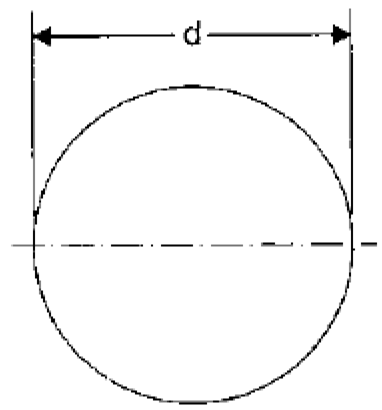
Hence moment of resistance offered by the section is maximum when Z is maximum. Hence Z represents the strength of the section.

Section Modulus









Procedures for determining bending stresses Stress at a Given Point

1. Use the method of sections to determine the bending moment M at the cross section containing the given point.
2. Determine the location of the neutral axis.
3. Compute the moment of inertia I of the cross- sectional area about the neutral axis. (If the beam is standard structural shape, its cross- sectional properties are listed in Appendix B. P501)
4. Determine the y-coordinate of the given point. Note that y is positive if the point lies above the neutral axis and negative if it lies below the neutral axis.
5. Compute the bending stress from $\sigma = -My/I$. If correct sign are used for *M and y, the stress will also have the correct sign* (tension positive compression negative).

A steel wire of 5 mm diameter is bent into a circular shape of 5 m radius. Determine the maximum stress induced in the wire. Take E = 200 GPa.

SOLUTION. Given : Diameter of steel wire (d) = 5 mm ; Radius of circular shape (R) = 5 m = 5×10^3 mm and modulus of elasticity (E) = 200 GPa = 200×10^3 N/mm^2.

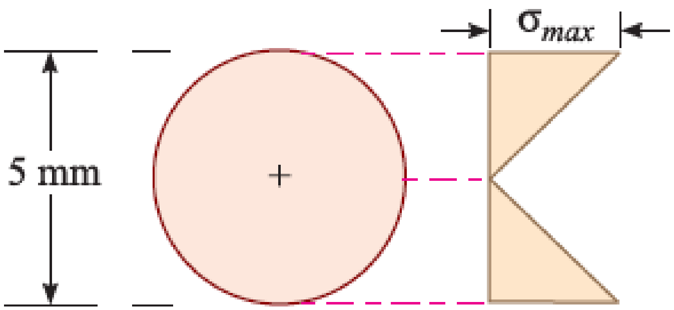


Fig. 14.3

We know that distance between the neutral axis of the wire and its extreme fibre,

$$y = \frac{d}{2} = \frac{5}{2} = 2.5 \text{ mm}$$

and maximum bending stress induced in the wire,

$$\sigma_{b\,(max)} = \frac{E}{R} \times y = \frac{200 \times 10^3}{5 \times 10^3} \times 2.5 = 100 \text{ N/mm}^2 = 100 \text{ MPa} \qquad \textbf{Ans.}$$

A copper wire of 2 mm diameter is required to be wound around a drum. Find the minimum radius of the drum, if the stress in the wire is not to exceed 80 MPa. Take modulus of elasticity for the copper as 100 GPa.

SOLUTION. Given : Diameter of wire (d) = 2 mm ; Maximum bending stress $\sigma_{b\,(max)} = 80\text{ MPa} = 80\text{ N/mm}^2$ and modulus of elasticity (E) = 100 GPa = $100 \times 10^3\text{ N/mm}^2$.

We know that distance between the neutral axis of the wire and its extreme fibre

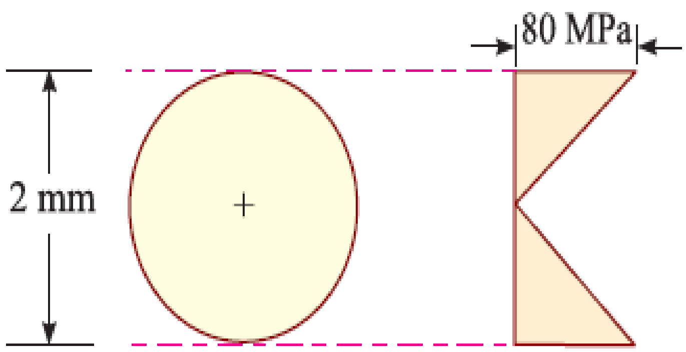


Fig. 14.4

$$y = \frac{2}{2} = 1\text{ mm}$$

∴ Minimum radius of the drum

$$R = \frac{y}{\sigma_{b\,(max)}} \times E = \frac{1}{80} \times 100 \times 10^3 \qquad \ldots\left(\because \frac{\sigma_b}{y} = \frac{E}{R}\right)$$

$$= 1.25 \times 10^3\text{ mm} = 1.25\text{ m}$$ **Ans.**

A rectangular beam 60 mm wide and 150 mm deep is simply supported over a span of 6 m. If the beam is subjected to central point load of 12 kN, find the maximum bending stress induced in the beam section.

SOLUTION. Given : Width (b) = 60 mm ; Depth (d) = 150 mm ; Span (l) = 6×10^3 mm and load (W) = 12 kN = 12×10^3 N.

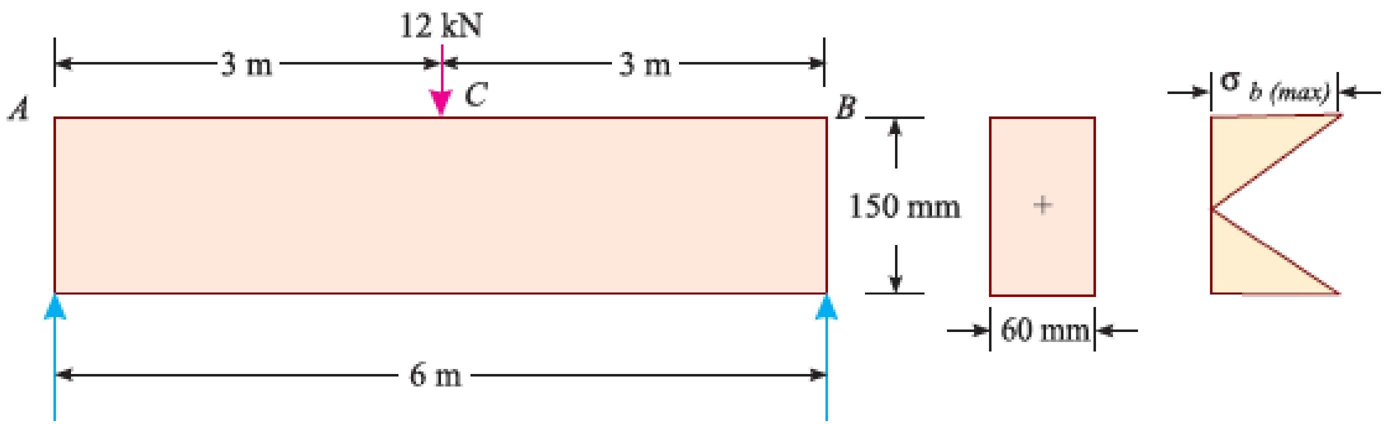


Fig. 14.15

We know that maximum bendint moment at the centre of a simply supported beam subjected to a central point load,

$$M = \frac{Wl}{4} = \frac{(12 \times 10^3) \times (6 \times 10^3)}{4} = 18 \times 10^6 \text{ N-mm}$$

and section modulus of the rectangular section,

$$Z = \frac{bd^2}{6} = \frac{60 \times (150)^2}{6} = 225 \times 10^3 \text{ mm}^3$$

∴ Maximum bending stress,

$$\sigma_{max} = \frac{M}{Z} = \frac{18 \times 10^6}{225 \times 10^3} = 80 \text{ N/mm}^2 = 80 \text{ MPa} \qquad \textbf{Ans.}$$

Two wooden planks 150 mm × 50 mm each are connected to form a T-section of a beam. If a moment of 6.4 kN-m is applied around the horizontal neutral axis, inducing tension below the neutral axis, find the bending stresses at both the extreme fibres of the cross- section.

Solution. Given: Size of wooden planks = 150 mm × 50 mm and moment (M) = 6.4 kN-m = 6.4 × 10^6 N-mm.

Two planks forming the T-section are shown in Fig. 15.1. First of all, let us find out the centre of gravity of the beam section. We know that distance between the centre of gravity of the section and its bottom face,

$$\bar{y} = \frac{(150 \times 50)\,175 + (150 \times 50)\,75}{(150 \times 50) + (150 \times 50)} = \frac{1875000}{15000} = 125 \text{ mm}$$

∴ Distance between the centre of gravity of the section and the upper extreme fibre,

$$y_t = 20 - 125 = 75 \text{ mm}$$

and distance between the centre of gravity of the section and the lower extreme fibre,

$$y_c = 125 \text{ mm}$$

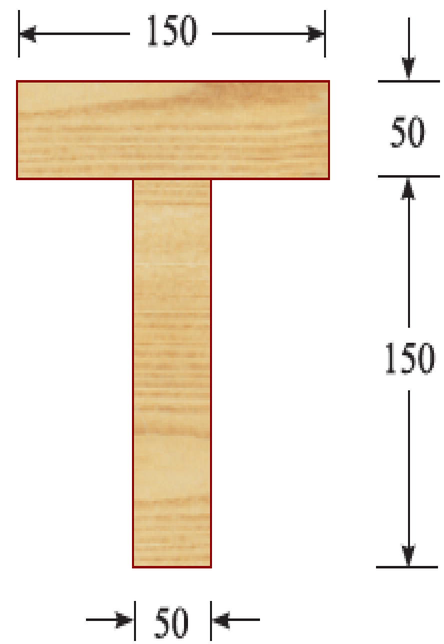
We also know that Moment of inertia of the T section about an axis passing through its c.g. and parallel to the botom face,

$$I = \left[\frac{150\times(50)^3}{12} + (150\times50)\,(175-125)^2\right] + \left[\frac{50\times(150)^3}{12} + (150\times50\,(125-75)^2\right] \text{mm}^4$$

$$= (20.3125 \times 10^6) + (32.8125 \times 10^6)\ \text{mm}^4$$

$$= 53.125 \times 10^6\ \text{mm}^4$$

∴ Bending stress in the upper extreme fibre,



$$\sigma_1 = \frac{M}{I} \times y_t = \frac{6.4\times10^6}{53.125\times10^6} \times 125\ \text{N/mm}^2$$

$$= 15.06\ \text{N/mm}^2 = 15.06\ \text{MPa (compression)}$$ **Ans.**

and bending stress in the lower extreme fibre,

$$\sigma_2 = \frac{M}{I} \times y_c = \frac{6.4\times10^6}{53.125\times10^6} \times 75\ \text{N/mm}^2$$

$$= 9.04\ \text{N/mm}^2 = 9.04\ \text{MPa (tension)}$$ **Ans.**

Figure 15.2 shows a rolled steel beam of an unsymmetrical I-section.

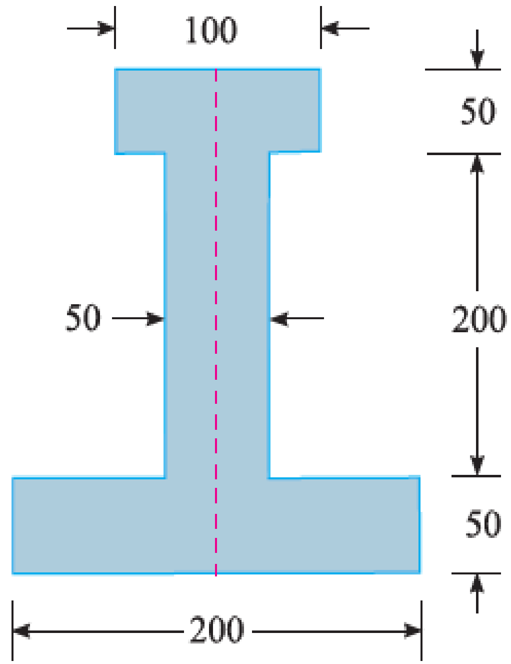


Fig. 15.2

If the maximum bending stress in the beam section is not to exceed 40 MPa, find the moment, which the beam can resist.

SOLUTION. Given: Maximum bending stress (σ_{max}) = 40 MPa = 40 N/mm^2.

SOLUTION. Given: Maximum bending stress (σ_{max}) = 40 MPa = 40 N/mm^2.

We know that distance between the centre of gravity of the section and bottom face,

$$\bar{y} = \frac{(100 \times 50)\, 275 + (200 \times 50)\, 150 + (200 \times 50)\, 25}{(100 \times 50) + (200 \times 50) + (200 \times 50)} = 125 \text{ mm}$$

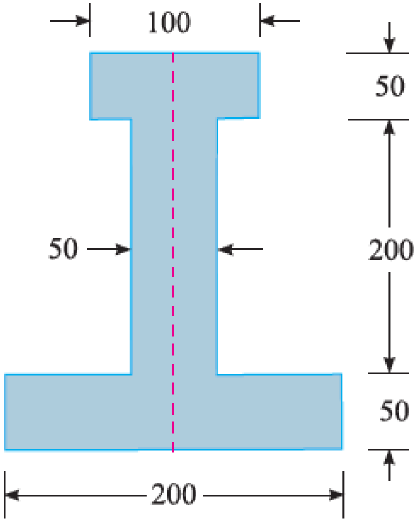
$\therefore \quad y_1 = 300 - 125 = 175$ mm $\quad$ and $\quad y_2 = 125$ mm

Thus we shall take the value of $y = 175$ mm (*i.e.*, greater of the two values between y_1 and y_2). We also know that moment of inertia of the *I*-section about an axis passing through its centre of gravity and parallel to the bottom face,

$$I = \left[\frac{100 \times (50)^3}{12} + (100 \times 50)\,(275 - 125)^2\right] + \left[\frac{50 \times (200)^3}{12} + (50 \times 200)\,(150 - 125)^2\right]$$

$$+ \left[\frac{200 \times (50)^3}{12} + (200 \times 50)\,(125 - 25)^2\right] \text{mm}^4$$

$$= 255.2 \times 10^6 \text{ mm}^4$$



and section modulus of the *I*-section,

$$Z = \frac{I}{y} = \frac{255.2 \times 10^6}{175} = 1.46 \times 10^6 \text{ mm}^3$$

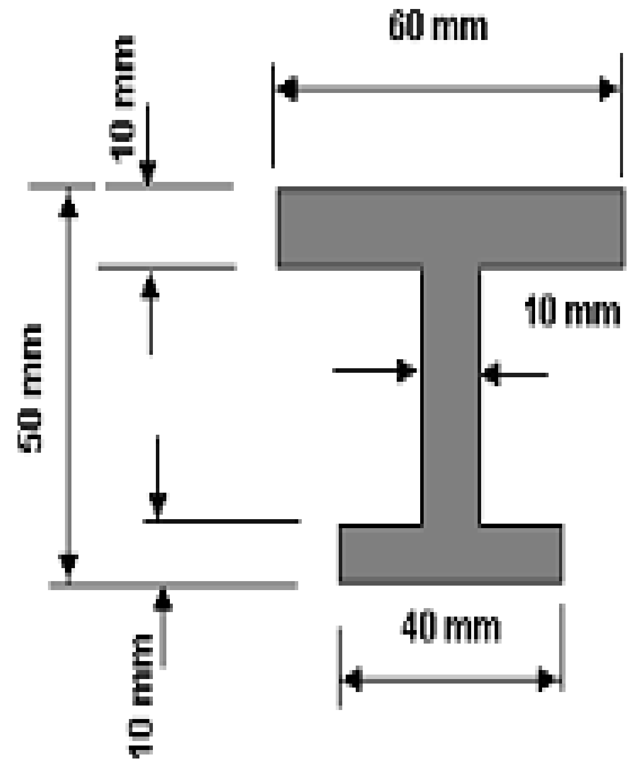
$\therefore$ Moment, which the beam can resist,

$$M = \sigma_{max} \times Z = 40 \times (1.46 \times 10^6) \text{ N-mm}$$

$$= 58.4 \times 10^6 \text{ N-mm} = 58.4 \text{ kN-m}$$ **Ans.**

Example 1: I-Section

Calculate the stress on the top and bottom of the section shown when the bending moment is 300 N m. Draw the stress distribution.



Now calculate the stress using the well known formula $\sigma_B = My/I$

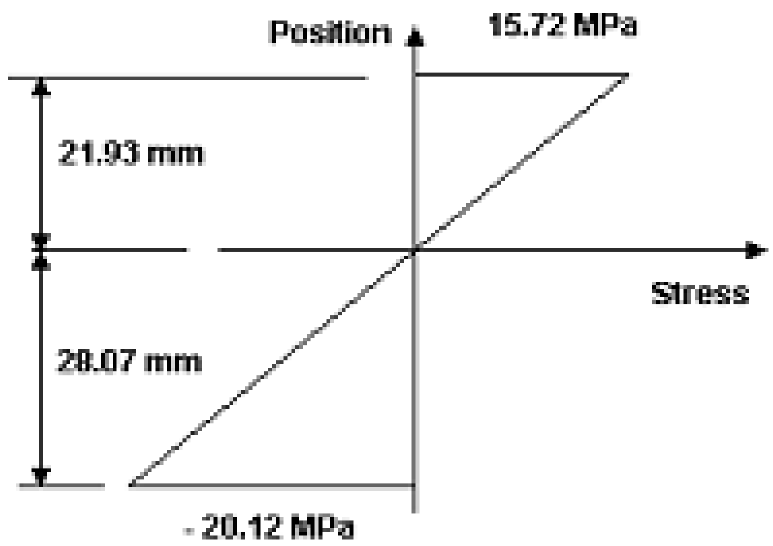
Top edge y = distance from the centroid to the edge = 50 – 28.08 = 21.93 mm

$\sigma_B = 300 \times 0.02192/418.300 \times 10^{-9} = 15.72 \times 10^{6}$ Pa or 15.72 MPa (Tensile)

Bottom edge y = $\bar{y}$=28.07 mm

$\sigma_B = 300 \times 0.02808/418.300 \times 10^{-9} = 20.14$ MPa (Tensile)

The stress distribution looks like this.



Example 2:

A beam has a hollow circular cross section 40 mm outer diameter and 30 mm inner diameter. It is made from metal with a modulus of elasticity of 205 GPa. The maximum tensile stress in the beam must not exceed 350 MPa.

Calculate the following.

(i) the maximum allowable bending moment.
(ii) the radius of curvature.

SOLUTION

D = 40 mm, d = 30 mm

$I = \pi(40^4 - 30^4)/64 = 85.9 \times 10^3\ mm^4$ or $85.9 \times 10^{-9}\ m^4$.

The maximum value of y is D/2 so y = 20 mm or 0.02 m

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$M = \frac{\sigma I}{y} = \frac{350 \times 10^6 \times 85.9 \times 10^{-9}}{0.02} = 1503\ Nm \text{ or } 1.503\ M\,Nm$$

$$\frac{\sigma}{y} = \frac{E}{R}$$

$$R = \frac{Ey}{\sigma} = \frac{205 \times 10^9 \times 0.02}{350 \times 10^6} = 11.71\ m$$

Example 3:

The section solved in example 2 is subjected to a tensile force that adds a tensile stress of 10 MPa everywhere. Sketch the stress distribution and determine the new position of the neutral axis.

SOLUTION

The stress on the top edge will increase to 25.72 MPa and on the bottom edge it will decrease to -10.12 MPa. The new distribution will be as shown and the new position of the neutral axis may be calculated by ratios.

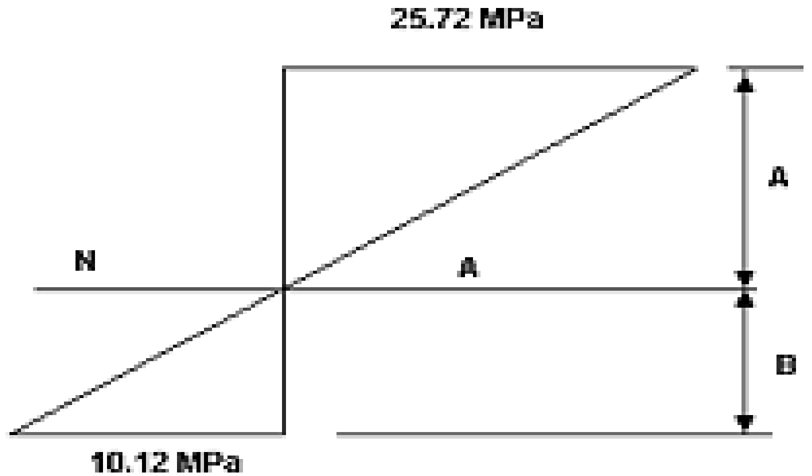


Figure 11

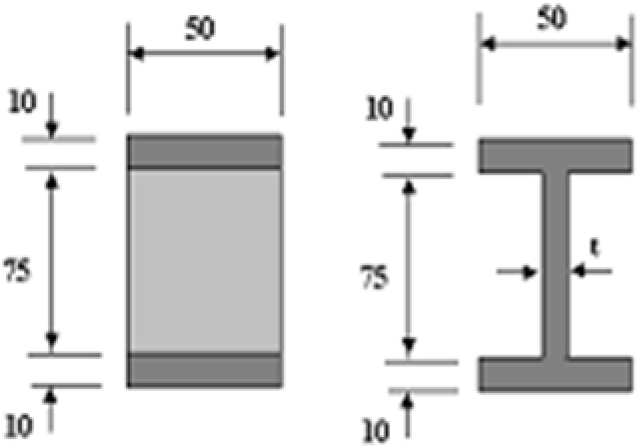
A + B = 50 mm   so B = 50 - A

By similar triangles   A/25.72 = B/10.12   A = (25.72/10.12)B = 2.54 B

B = 50 - 2.54 B   3.54 B = 50   B = 14.12 mm   A= 50 - 14.12 = 35.88 mm

Example 4:

A rectangular section timber beam is 50 mm wide and 75 mm deep. It is clad with steel plate 10 mm thick on the top and bottom. Calculate the maximum stress in the steel and the timber when a moment of 4 kNm is applied.
E for timber is 10 GPa and for steel 200 GPa



SOLUTION

The width of an equivalent steel web must be
$t = 50 \times E_t / E_s = 50 \times 10/200 = 2.5$ mm
Now calculate I_{gg} for the equivalent beam.
This is easy because it is symmetrical and involves finding I for the outer box and subtracting I for the missing parts.
$I_{gg} = 50 \times 95^3/12 - 47.5 \times 75^3/12$
$I_{gg} = 1.9025 \times 10^6\ m^4$

The stress at y = 37.5 mm $\sigma = My/I = 4000 \times 0.0375/1.9025 \times 10^{-6} = 78.845$ MPa
The stress in the timber at this level will be different because of the different E value.
$\sigma_t = \sigma_s E_t/E_s = 3.942$ MPa
The stress at y = 47.5 mm will be the stress at the edge of the steel.
$\sigma_s = My/I = 4000 \times 0.0475/1.9025 \times 10^{-6} = 99.87$ MPa

SOLUTION

First calculate the second moment of area using the tabular method that you should already know. Divide the shape into three sections A, B and C. First determine the position of the centroid from the bottom edge.

	Area	$\bar{y}$	$A\bar{y}$
A	600 mm^2	45 mm	27 000 mm^3
B	300 mm^2	25 mm	7500 mm^3
C	400 mm^2	5 mm	2000 mm^3
Totals	1300 mm^2		365000 mm^3

For the whole section the centroid position is $\bar{y}$=365000/1300 = 28.07 mm

Now find the second moment of area about the base. Using the parallel axis theorem.

	$BD^3/12$	$A\bar{y}^2$	$I = BD^3/12 + A\bar{y}^2$
A	$60 \times 10^3/12$=5000 mm^4	600×45^2=1215000	1220000 mm^4
B	$10 \times 30^3/12$=22500 mm^4	300×25^2=187500	210000 mm^4
C	$40 \times 10^3/12$=3333 mm^4	400×5^2=10000	13333 mm^4
			Total = 1443333 mm^4

The total second moment of area about the bottom is 1443333 mm4

Now move this to the centroid using the parallel axis theorem.

$$I = 1443333 - A\bar{y}^2 = 1443333 - 1300 \times 28.08^2 = 418300 \text{ mm}^4$$

. Shearing Stress at a Section in a Loaded Beam

Consider a small portion *ABDC* of length *dx* of a beam loaded with uniformly distributed load as shown in Fig. 16.1 (*a*).

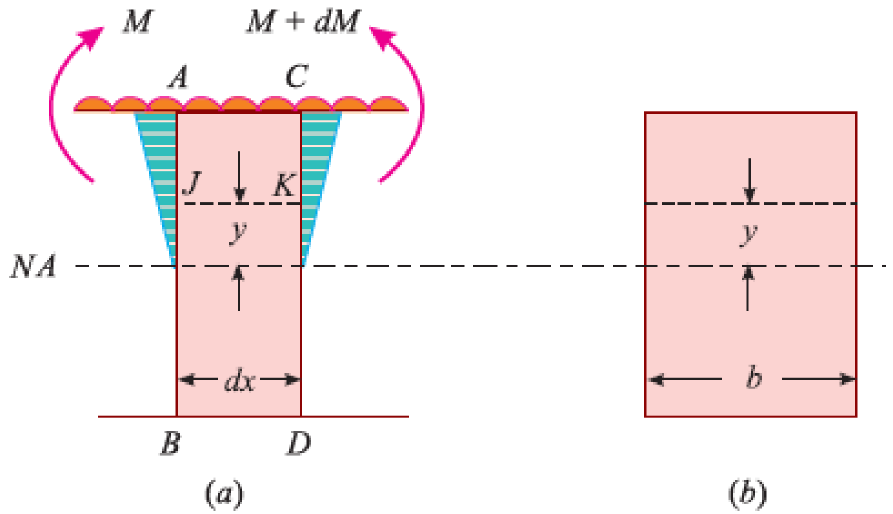


Fig. 16.1. Shearing stress

We know that when a beam is loaded with a uniformly distributed load, the shear force and bending moment vary at every point along the length of the beam.

Let M = Bending moment at *AB*,

$M + dM$ = Bending moment at *CD*,

$$F = \text{Shear force at } AB,$$

$$F + dF = \text{Shear force at } CD, \text{ and}$$

$$I = \text{Moment of inertia of the section about its neutral axis.}$$

Now consider an elementary strip at a distance y from the neutral axis as shown in Fig. 16.1 (*b*).

Now let σ = Intensity of bending stress across AB at distance y from the neutral axis and

a = Cross-sectional area of the strip.

We have already discussed that

$$\frac{M}{I} = \frac{\sigma}{y} \quad \text{or} \quad \sigma = \frac{M}{I} \times y \qquad \text{... (See Art. 14.6)}$$

Similarly, $$\sigma + d\sigma = \frac{M + dM}{I} \times y$$

where $\sigma + d\sigma$ = Intensity of bending stress across CD.

We know that the force acting across AB

$$= \text{Stress} \times \text{Area} = \sigma \times a = \frac{M}{I} \times y \times a \qquad \text{...}(i)$$

Similarly, force acting across CD

$$= (\sigma + d\sigma) \times a = \frac{M + dM}{I} \times y \times a \qquad \text{...}(ii)$$

$\therefore$ Net unbalanced force on the strip

$$= \frac{M + dM}{I} \times y \times a - \frac{M}{I} \times y \times a = \frac{dM}{I} \times y \times a$$

The total *unbalanced force (F) above the neutral axis may be found out by integrating the above equation between 0 and $d/2$.

or $$= \int_0^{\frac{d}{2}} \frac{dM}{I} a \cdot y \cdot dy = \frac{dM}{I} \int_0^{\frac{d}{2}} a \cdot y \cdot dy = \frac{dM}{I} A\bar{y} \qquad \text{...(iii)}$$

where A = Area of the beam above neutral axis, and $\bar{y}$ = Distance between the centre of gravity of the area and the neutral axis.

We know that the intensity of the shear stress,

$$\tau = \frac{\text{Total force}}{\text{Area}} = \frac{\frac{dM}{I} \cdot A\bar{y}}{dx \cdot b} \qquad \text{...(Where } b \text{ is the width of beam)}$$

$$= \frac{dM}{dx} \times \frac{A \cdot \bar{y}}{Ib}$$

$$= F \times \frac{A\bar{y}}{Ib} \qquad \left(\text{Substituting } \frac{dM}{dx} = F = \text{Shear force}\right)$$

An I-sections, with rectangular ends, has the following dimensions:

Flanges =150 mm × 20 mm, Web = 300 mm 10 mm.

Find the maximum shearing stress developed in the beam for a shear force of 50 kN.

SOLUTION. Given: Flange width (B) = 150 mm ; Flange thickness = 20 mm ; Depth of web (d) = 300 mm; Width of web = 10 mm; Overall depth of the section (D) = 340 mm and shearing force (F) = 50 kN = 50×10^3 N.

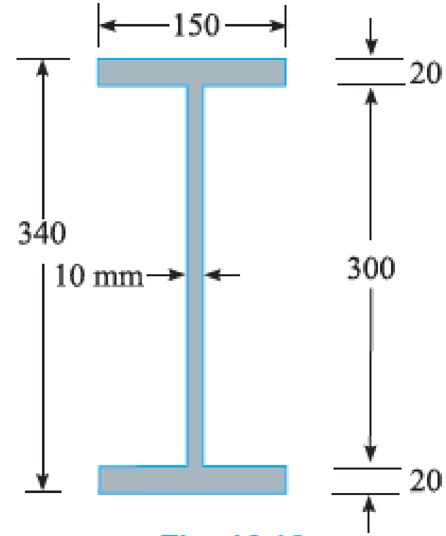


Fig. 16.10

We know that moment of inertia of the I-section about its centre of gravity and parallel to x-x axis,

$$I_{XX} = \frac{150 \times (340)^3}{12} - \frac{140 \times (300)^3}{12} \text{ mm}^4$$

$$= 176.3 \times 10^6 \text{ mm}^4$$

and maximum shearing stress,

$$\tau_{max} = \frac{F}{Ib}\left[\frac{B}{8}(D^2 - d^2) + \frac{bd^2}{8}\right]$$

$$= \frac{50 \times 10^3}{(176.3 \times 10^6) \times 10}\left[\frac{150}{8}[(340)^2 - (300)^2] + \frac{10 \times (300)^2}{8}\right] \text{N/mm}^2$$

$= 16.8$ N/mm^2 = **16.8 MPa** **Ans.**

An I-section beam 350 mm × 200 mm has a web thickness of 12.5 mm and a flange thickness of 25 mm. It carries a shearing force of 200 kN at a section. Sketch the shear stress distribution across the section.

SOLUTION. Given: Overall depth (D) = 350 mm ; Flange width (B) = 200 mm ; Width of Web = 12.5 mm ; Flange thickness = 25 mm and the shearing force (F) = 200 kN = 200×10^3 N.

We know that moment of inertia of the *I*-section about it centre of gravity and parallel to *x-x* axis,

$$I_{XX} = \frac{200 \times (350)^3}{12} - \frac{187.5 \times (300)^3}{12} = 292.7 \times 10^6 \text{ mm}^4$$

We also know that shear stress at the upper edge of the upper flange is zero. And shear stress at the joint of the upper flange and web

$$= \frac{F}{8I}[D^2 - d^2] = \frac{200 \times 10^3}{8 \times (292.7 \times 10^6)}[(350)^2 - (300)^2] \text{ N/mm}^2$$

$$= 2.78 \text{ N/mm}^2 = 2.78 \text{ MPa}$$

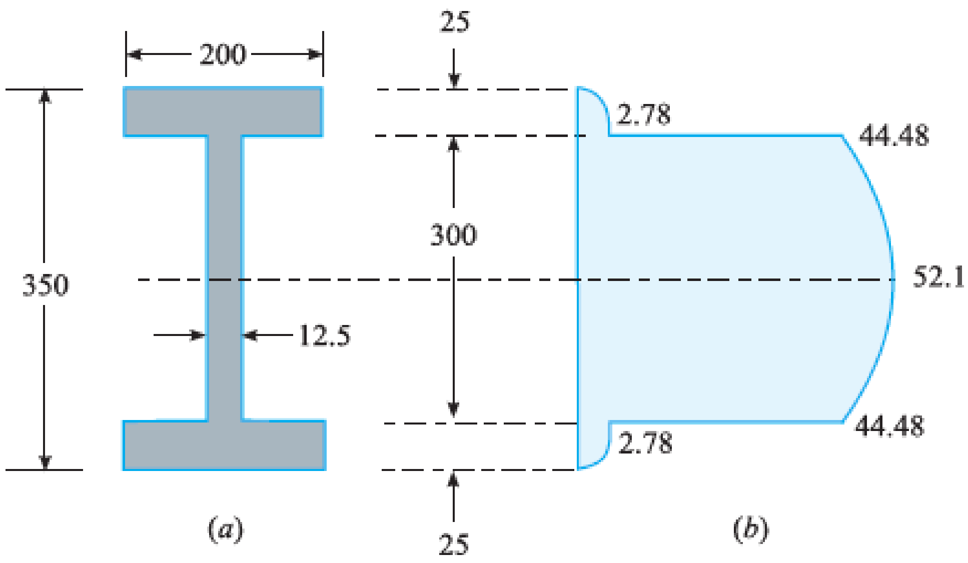


Fig. 16.11

The shear stress at the junction suddenly increases from 2.78 MPa to $2.78 \times \dfrac{200}{12.5} = 44.48$ MPa.

We also know that the maximum shear stress,

$$\tau_{max} = \frac{F}{I \cdot b}\left[\frac{B}{8}(D^2 - d^2) + \frac{bd^2}{8}\right]$$

$$= \frac{200 \times 10^3}{(292.7 \times 10^6) \times 12.5}\left[\frac{200}{8}(350)^2 - (300)^2 + \frac{12.5 \times (300)^2}{8}\right]$$

$$= 52.1 \text{ N/mm}^2 = 52.1 \text{ MPa}$$

Now complete the shear stress distribution diagram across the section as shown in Fig

A T-shaped cross-section of a beam shown in Fig. 16.12 is subjected to a vertical shear force of 100 kN. Calculate the shear stress at important points and draw shear stress distribution diagram. Moment of inertia about the horizontal neutral axis is mm⁴.

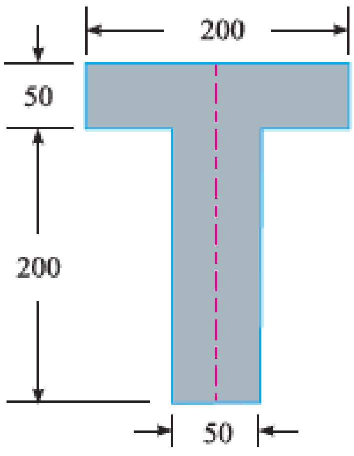


Fig. 16.12

SOLUTION. Given: Shear stress (F) = 100 kN = 100×10^3 N and moment of inertia (I) = 113.4×10^6 mm^4.

First of all let us find out the position of the neutral axis. We know that distance between the centre of gravity of the section and bottom of the web,

$$\bar{y} = \frac{[(200 \times 50) \times 225] + [(200 \times 50) \times 100]}{(200 \times 50) + (20 \times 50)}$$

$$= 162.5 \text{ mm}$$

∴ Distance between the centre of gravity of the section and top of the flange,

$$y_C = (200 + 50) - 162.5 = 87.5 \text{ mm}$$

We know that shear stress at the top of the flanges is zero. Now let us find out the shear stress at the junction of the flange and web by considering the area of the *flange of the section. We know that area of the upper flange,

$$A = 200 \times 50 = 10000 \text{ mm}^2$$

$$\bar{y} = 87.5 - \frac{50}{2} = 62.5 \text{ mm}$$

$$B = 200 \text{ mm}$$

∴ Shear stress at the junction of the flange and web,

$$\tau = F \times \frac{A \cdot \bar{y}}{I \cdot B} = 100 \times 10^3 \times \frac{10000 \times 62.5}{(113.4 \times 10^6) \times 200} \text{ N/mm}^2$$

$$= 2.76 \text{ N/mm}^2 = 2.76 \text{ MPa}$$

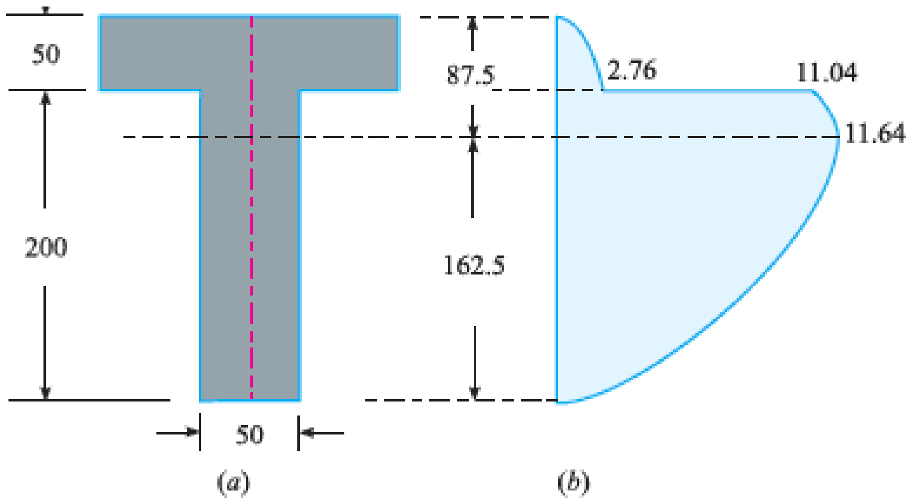


Fig. 16.13

The shear stress at the junction suddenly increases from 2.76 MPa to $2.76 \times \dfrac{200}{50} = 11.04$ MPa.

Now let us find out the shear stress at the neutral axis, where the shear stress is maximum.

Considering the area of the *T*-section above the neutral axis of the section, we know that

$$* A\bar{y} = [(200 \times 50) \times 62.5] + \left[(37.5 \times 50) \times \frac{37.5}{2}\right] \text{mm}^3$$

$$= 660.2 \times 10^3 \text{ mm}^3$$

and $b = 50$ mm

∴ Maximum shear stress,

$$\tau_{max} = F \times \frac{A \cdot \bar{y}}{I \cdot b} = 100 \times 10^3 \times \frac{660.2 \times 10^3}{(113.4 \times 10^6) \times 50} \text{ N/mm}^2$$

$$= 11.64 \text{ N/mm}^2 = 11.64 \text{ MPa}$$

A cast-iron bracket subjected to bending, has a cross-section of I-shape with unequal flanges as shown in Fig. 16.14.

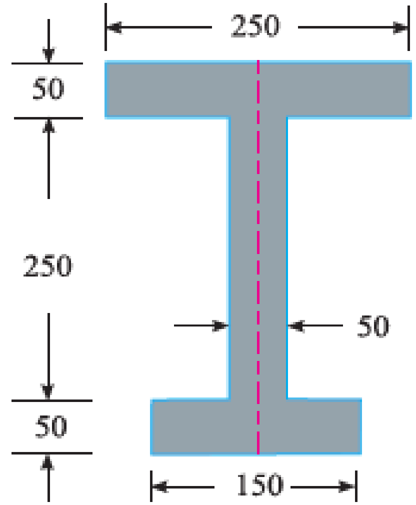


Fig. 16.14

If the compressive stress in top flange is not to exceed 17.5 MPa, what is the bending moment, the section can take? If the section is subjected to a shear force of 100 kN, draw the shear stress distribution over the depth of the section.

SOLUTION. Given: Compressive stress (σ_c) = 17.5 MPa = 17.5 N/mm^2 and shear force (F) = 100 kN = 100×10^3 N

Bending moment the section can take

First of all, let us find out the position of the neutral axis. We know that distance between centre of gravity of the section and bottom face,

$$\bar{y} = \frac{(250 \times 50)\ 325 + (250 \times 50)\ 175 + (150 \times 50)\ 25}{(250 \times 50) + (250 \times 50) + (150 \times 50)}$$

$$= \frac{6\,437\,500}{32\,500} = 198 \text{ mm}$$

∴ Distance of centre of gravity from the upper extreme fibre,

$$y_c = 350 - 198 = 152 \text{ mm}$$

and moment of inertia of the section about an axis passing through its centre of gravity and parallel to *x-x* axis,

$$I = \left[\frac{250 \times (50)^3}{12} + (250 \times 50)\,(325 - 198)^2\right]$$

$$+ \left[\frac{50 \times (250)^3}{12} + (50 \times 250)\,(198 - 175)^2\right]$$

$$+ \left[\frac{150 \times (50)^3}{12} + (150 \times 50)\,(198 - 25)^2\right] \text{mm}^4$$

$$= 502 \times 10^6 \text{ mm}^4$$

∴ Bending moment the section can take

$$= \frac{\sigma_c}{y_c \times I} = \frac{17.5}{152} \times 502 \times 10^6 = 57.8 \times 10^6 \text{ N-mm}$$

$$= 57.8 \text{ kN-m}$$ **Ans.**

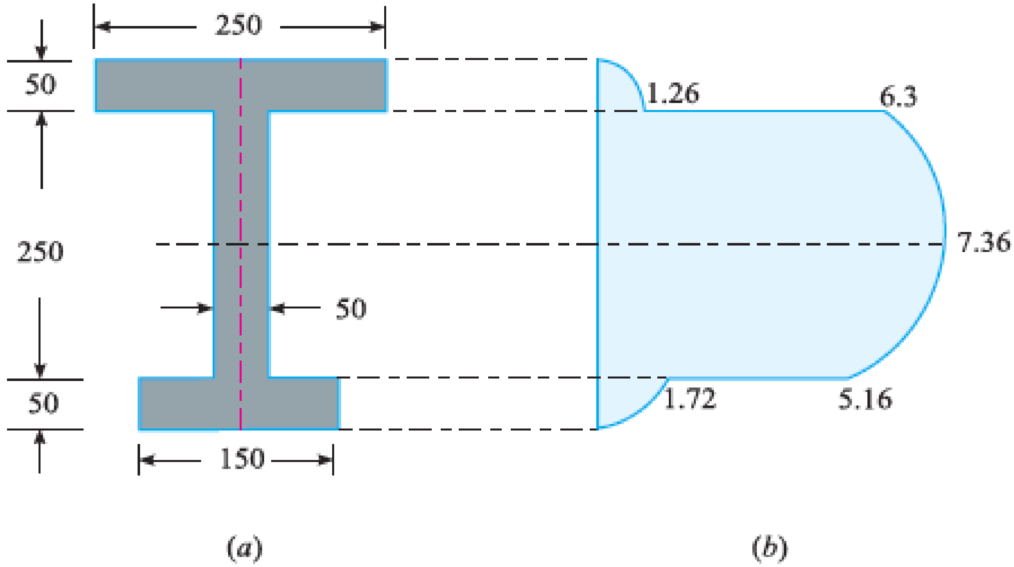


Fig. 16.15

$$A = 250 \times 50 = 12500 \text{ mm}$$

$$\bar{y} = 152 - \frac{50}{2} = 127 \text{ mm}$$

and $$B = 250 \text{ mm}$$

∴ Shear stress at the junction of the upper flange and web,

$$\tau = F \times \frac{A \cdot \bar{y}}{I \cdot B} = 100 \times 10^3 \frac{12500 \times 127}{(502 \times 10^6) \times 250} \text{ N/mm}^2$$

$$= 1.26 \text{ N/mm}^2 = 1.26 \text{ MPa}$$

The shear stress at the junction suddenly increases from 1.26 MPa to $1.26 \times \frac{250}{50} = 6.3$ MPa.

The shear stress at the junction suddenly increases from 1.26 MPa to $1.26 \times \frac{250}{50} = 6.3$ MPa.

Now let us find out the shear stress at the junction of the lower flange and web by considering the area of the lower flange. We know that area of the lower flange,

$$A = 150 \times 50 = 7500 \text{ mm}^2$$

$$\bar{y} = 198 - \frac{50}{2} = 173 \text{ mm}$$

and $$B = 150 \text{ mm}$$

∴ Shear stress at the junction of the lower flange and web,

$$\tau = F \times \frac{A \cdot \bar{y}}{I \cdot B} = 100 \times 10^3 \times \frac{7500 \times 173}{(502 \times 10^6) \times 150}$$

$$= 1.72 \text{ N/mm}^2 = 1.72 \text{ MPa}$$

The shear stress at the function suddenly increases from 1.72 MPa to $1.72 \times \frac{150}{50} = 5.16$ MPa.

Now let us find out the shear stress at the neutral axis, where the shear stress is maximum. Considering the area of the *I*-section above neutral axis, we know that

$$A\bar{y} = [(250 \times 50) \times 127] + \left[(102 \times 50) \times \frac{102}{2}\right] \text{mm}^3$$

$$= 1.848 \times 106 \text{ mm}^3$$

and $$b = 50 \text{ mm}$$

∴ Maximum shear stress,

$$\tau_{max} = F \times \frac{A \cdot \bar{y}}{I \cdot b} = 100 \times 10^3 \times \frac{1.848 \times 10^6}{(502 \times 10^6) \times 50} \text{ N/mm}^2$$

$$= 7.36 \text{ N/mm}^2 = 7.36 \text{ MPa}$$

Now draw the shear stress distribution diagram over the depth of the section as shown in Fig. 16.15.

Distribution of Shearing Stress over a Rectangular Section

Consider a beam of rectangular section *ABCD* of width and depth as shown in Fig. 16.2 (*a*). We know that the shear stress on a layer *JK* of beam, at a distance *y* from the neutral axis,

$$\tau = F \times \frac{A\bar{y}}{Ib} \qquad \ldots(i)$$

* This may also be found out by splitting up the beam into number of strips at distance of from the neutral axis.

We know that unbalanced force on strip 1 $= \frac{dM}{I} \times a_1 \cdot y_1$

Similarly, unbalanced force on strip 2 $= \frac{dM}{I} \times a_2 \cdot y_2$

and unbalanced force on strip 3 $= \frac{dM}{I} \times a_3 \cdot y_3$ and so on

$\therefore$ Total force, $F = \frac{dM}{I} \times a_1 \cdot y_1 + \frac{dM}{I} \times a_2 \cdot y_2 + \frac{dM}{I} \times a_3 \cdot y_3 + \ldots.$

$= \frac{dM}{I}(a_1 . y_1 + a_2 . y_2 + a_3 . y_3 + \ldots) = \frac{dM}{I} A\bar{y}$

where F = Shear force at the section,

A = Area of section above y (*i.e.*, shaded area *AJKD*),

$\bar{y}$ = Distance of the shaded area from the neutral axis,

$\therefore$ $A\bar{y}$ = Moment of the shaded area about the neutral axis,

I = Moment of inertia of the whole section about its neutral axis, and

b = Width of the section.

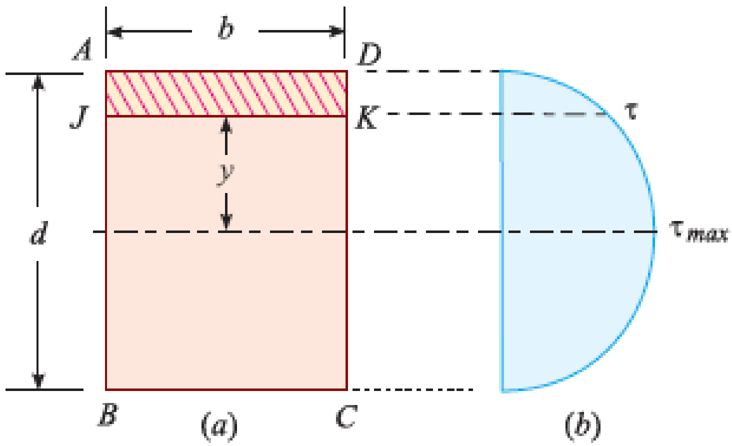


Fig. 16.2. Rectangular section

We know that area of the shaded portion *AJKD*,

$$A = b\left(\frac{d}{2} - y\right) \qquad \text{...(ii)}$$

$$\therefore \quad \bar{y} = y + \frac{1}{2}\left(\frac{d}{2} - y\right) = y + \frac{d}{4} - \frac{y}{2}$$

$$= \frac{y}{2} + \frac{d}{4} = \frac{1}{2}\left(y + \frac{d}{2}\right) \qquad ...(iii)$$

Substituting the above values of A and $\bar{y}$ in equation (i),

$$\tau = F \times \frac{A\bar{y}}{Ib} = F \times \frac{b\left(\frac{d}{2} - y\right) \times \frac{1}{2}\left(y + \frac{d}{2}\right)}{Ib}$$

$$= \frac{F}{2I}\left(\frac{d^2}{4} - y^2\right) \qquad ...(iv)$$

We see, from the above equation, that τ increase as y decreases. At a point, where $y = d/2$, $\tau = 0$; and where y is zero, τ is maximum. We also see that the variation of τ with respect to y is a parabola.

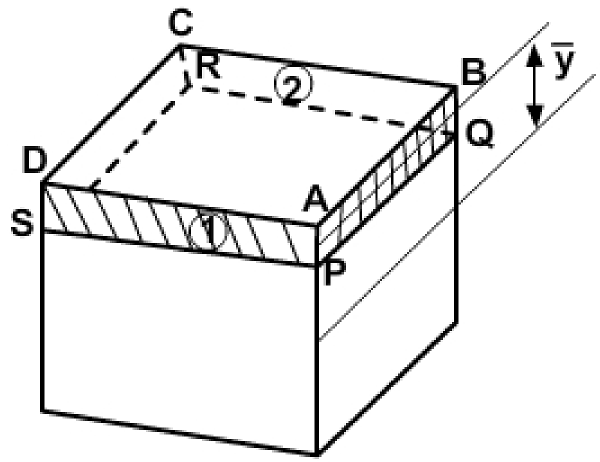
At neutral axis, the value of τ is maximum. Thus substituting $y = 0$ and $I = \frac{bd^3}{12}$ in the above equation,

$$\tau_{max} = \frac{F}{2 \times \frac{ba^3}{12}}\left(\frac{d^2}{4}\right) = \frac{3F}{2bd} = 1.5\,\tau_{av} \qquad ...\left(\because \tau_{av} = \frac{F}{\text{Area}} = \frac{F}{bd}\right)$$

Now draw the shear stress distribution diagram as shown in Fig. 16.2 (b).

Shear Stresses

Shear force is related to change in bending moment between adjacent sections.



Cut-out section from a beam

Problem 1: Derivation of Shear stress in rectangular cross-section

Derive an expression for the shear stress distribution in a beam of solid rectangular cross-section transmitting a vertical shear V.

A longitudinal cut through the beam at a distance y_1 from the neutral axis, isolates area klmn. (A_1).

Shear stress,

$$\tau = \frac{VQ}{It}$$

$$= \frac{V}{It}\int_{A_1} y.dA$$

$$= \frac{V}{Ib}\int_{y_1}^{d/2} by\,dy$$

$$= \frac{V}{2I}\left[(d/2)^2 - (y_1)^2\right] \text{ ---------------------- (1)}$$

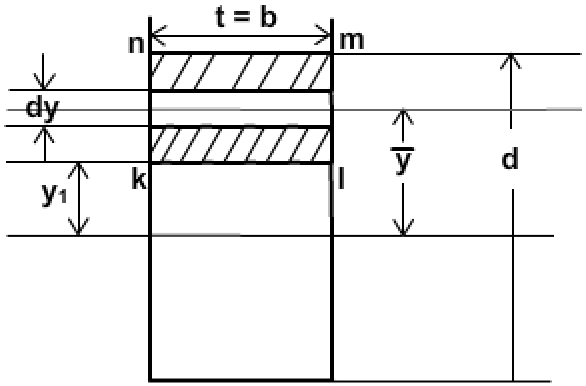


Fig. C/S area of the beam

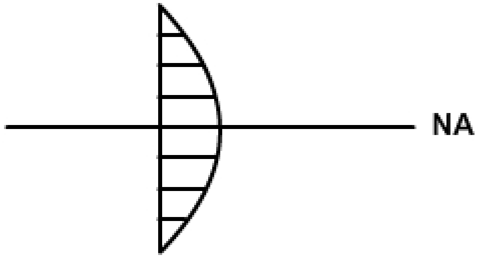


Fig. Shear Stress distribution

Problem 1: Derivation of Shear stress in rectangular cross-section contd. (Max.)

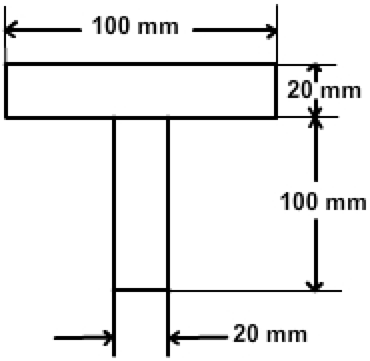
Max Shear Stress occurs at the NA and this can be found by putting y=0 in the Eq. (1).

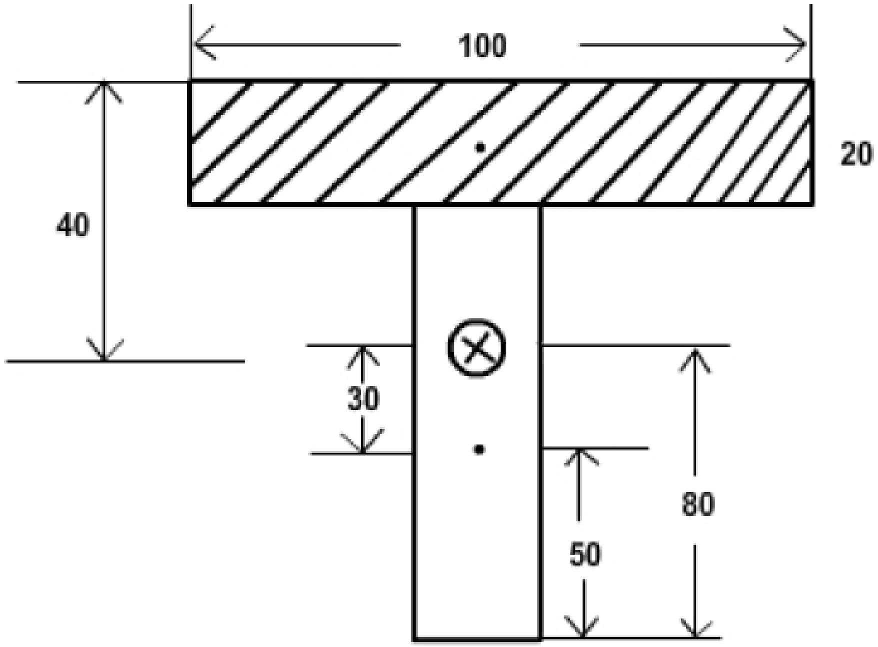
$$\tau_{max} = \frac{Vd^2}{8I}$$

$$= \frac{3}{2}\frac{V}{bh}$$

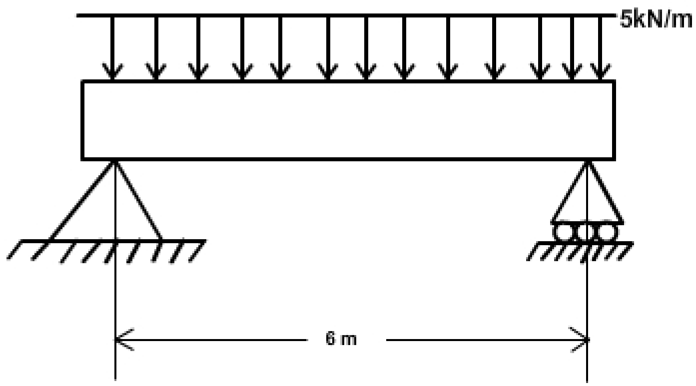
$$= \frac{3}{2}\frac{V}{A}$$

Problem 2: A vertical shear force of 1KN acts on the cross section shown below. Find the shear at the interface (per unit length)

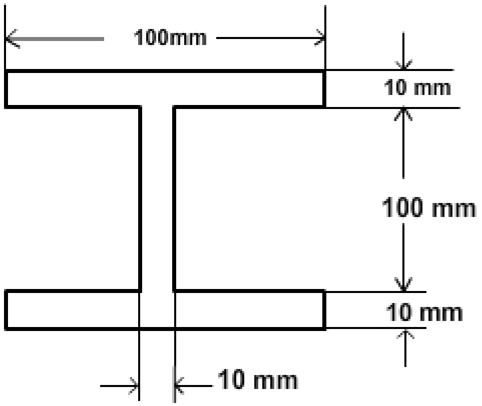


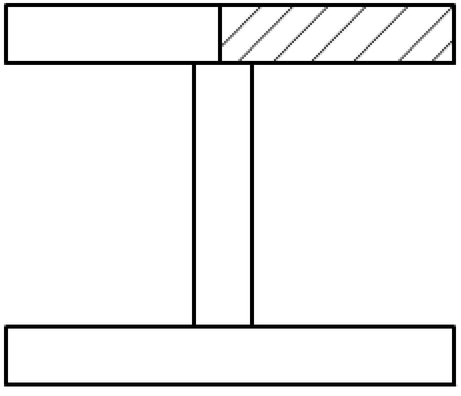


Problem 3: A 6m long beam with a 50 mm × 50 mm cross section is subjected to uniform loading of 5KN/m. Find the max shear stress in the beam



Problem 4: The cross section of an I beam is shown below. Find the max.shear stress in the flange if it transmits a vertical shear of 2KN.





Need

In all practical engineering applications, when we use the different components, normally we have to operate them within the certain limits

Constraints are placed on the performance and behavior of the components

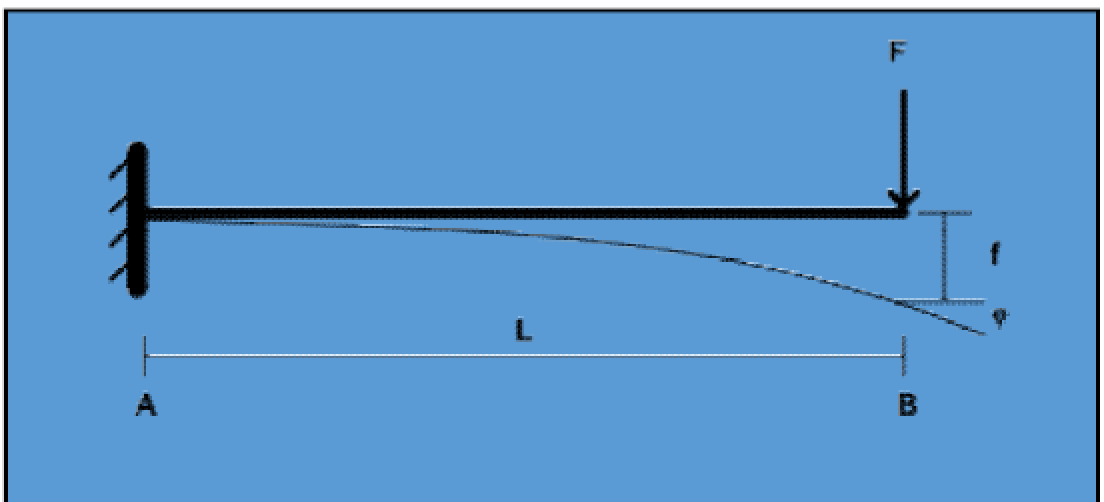
For instance we say that the particular component is supposed to operate within this value of stress and the deflection of the component should not exceed beyond a particular value

In some problems the maximum stress however, may not be a strict or severe condition but there may be the deflection which is the more rigid condition under operation

Deflection and Slope

Deflection is a term that is used to describe the degree to which a structural element is displaced under a load

Slope is the angle made by tangent drawn to deflected shape with the original shape



Elastic Curve

- The deflection diagram of the longitudinal axis that passes through the centre of each cross-sectional area of the beam
- Support that resist a force, such as **pinned, restrict displacement**
- Support that resist a moment such as **fixed, resist rotation or slope as well as displacement**

Relation Between B. M. And Slope

Assumption:

1. Stress is proportional to strain i.e. hooks law applies. Thus, the equation is valid only for beams that are not stressed beyond the elastic limit.
2. The curvature is always small.
3. Any deflection resulting from the shear deformation of the material or shear stresses is neglected.
4. Deflections due to shear deformations are usually small and hence can be ignored.

Relation between Slope, Deflection and Radius of Curvature

Consider a small portion PQ of a beam, bent into an arc as shown in Fig. 19.2.

Let ds = Length of the beam PQ,

R = Radius of the arc, into which the beam has been bent,

C = Centre of the arc,

Ψ = Angle, which the tangent at P makes with x-x axis and

$\Psi + d\Psi$ = Angle which the tangent at Q makes with x-x axis.

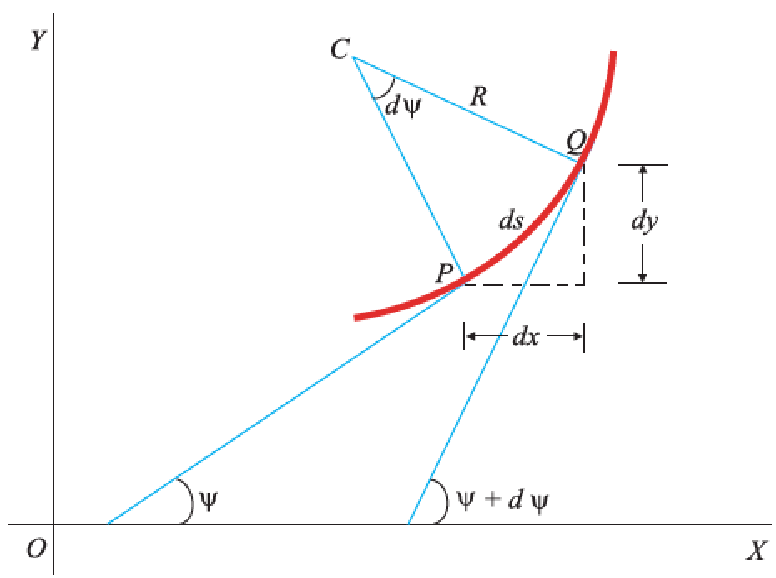
From the geometry of the figure, we find that

$$\angle PCQ = d\Psi$$

and

$$ds = R \, . \, d\Psi$$

$$\therefore \quad R = \frac{ds}{d\Psi} = \frac{dx}{d\Psi} \quad \text{... (Considering } ds = dx\text{)}$$



or $$\frac{1}{R} = \frac{d\Psi}{dx} \quad ...(i)$$

We know that if x and y be the co-ordinates of point P, then

$$\tan \Psi = \frac{dy}{dx}$$

Since Ψ is a very small angle, therefore taking $\tan \Psi = \Psi$,

$$\therefore \quad \frac{d\Psi}{dx} = \frac{d^2y}{dx^2} \quad ...\left(\because \frac{1}{R} = \frac{d\Psi}{\mathrm{d}x}\right)$$

We also know that

$$\frac{M}{I} = \frac{E}{R} \quad \text{or} \quad M = EI \times \frac{I}{R}$$

$$\therefore \quad M = EI \times \frac{d^2y}{dx^2} \quad ...\left(\text{Substituting value of } \frac{1}{R}\right)$$

NOTE. The above equation is also based only on the bending moment. The effect of shear force, being very small as compared to the bending moment, is neglected.

Simply Supported Beam with a Central Point Load

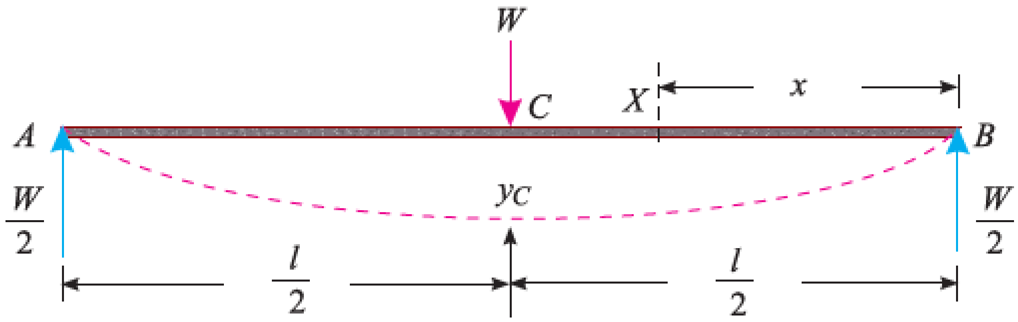


Fig. 19.3. Simply supported beam with a central point load.

Consider a simply supported beam *AB* of length *l* and carrying a point load *W* at the centre of beam *C* as shown in Fig. 19.3. From the geometry of the figure, we find that the reaction at *A*,

$$R_A = R_B = \frac{W}{2}$$

Consider a section X at a distance x from B. We know that the bending moment at this section,

$$M_X = R_B \cdot x = \frac{W}{2} x = \frac{Wx}{2} \qquad \text{... (Plus sign due to sagging)}$$

$$\therefore \qquad EI \frac{d^2y}{dx^2} = \frac{Wx}{2} \qquad \text{...}(i)$$

Integrating the above equation,

$$EI \frac{dy}{dx} = \frac{Wx^2}{4} + C_1 \qquad \text{...}(ii)$$

where C_1 is the first constant of integration. We know that when $x = \frac{l}{2}$, then $\frac{dy}{dx} = 0$. Substituting these values in equation (*ii*),

$$0 = \frac{Wl^2}{16} + C_1 \qquad \text{or} \qquad C_1 = -\frac{Wl^2}{16}$$

Substituting this value of C_1 in equation (*ii*),

$$El \frac{dy}{dx} = \frac{Wx^2}{4} - \frac{Wl^2}{16} \qquad \text{...}(iii)$$

This is the required equation for the slope, at any section. It will be interesting to know that the maximum slope occurs at A and B. Thus for maximum slope at B, substituting $x = 0$ in equation (*iii*),

$$El \cdot i_B = -\frac{Wl^2}{16}$$

$$\therefore \quad i_B = -\frac{Wl^2}{16EI}$$ ...(Minus sign means that the tangent at B makes an angle with AB in the negative or anticlockwise direction)

or
$$i_B = \frac{Wl^2}{16EI} \text{ radians}$$

By symmetry,
$$i_A = \frac{Wl^2}{16EI} \text{ radians}$$

Integrating the equation (*iii*) once again,

$$\therefore \quad E\,I.y = \frac{Wx^3}{12} - \frac{Wl^2 x}{16} + C_2 \quad \text{...(iv)}$$

where C_2 is the second constant of integration. We know that when $x = 0$, then $y = 0$, Substituting these values in equation (*iv*), we get $C_2 = 0$.

$$\therefore \quad E\,I.y = \frac{Wx^3}{12} - \frac{Wl^2 x}{16} \quad \text{...(v)}$$

This is the required equation for the deflection, at any section. A little consideration will show that maximum deflection occurs at the mid-point C. Thus for maximum deflection, substituting $x = \frac{l}{2}$ in equation (*v*),

$$EIy_C = \frac{W}{12}\left(\frac{l}{2}\right)^3 - \frac{Wl^2}{16}\left(\frac{l}{2}\right)$$

$$= \frac{Wl^3}{96} - \frac{Wl^3}{32} = -\frac{Wl^3}{48}$$

or
$$y_C = -\frac{Wl^3}{48EI}$$ ... (Minus sign means that the deflection is downwards)

$$= \frac{Wl^3}{48EI}$$

A simply supported beam of span 3 m is subjected to a central load of 10 kN. Find the maximum slope and deflection of the beam. Take $I = 12 \times 10^6$ mm^4 and $E = 200$ GPa.

SOLUTION. Given: Span (l) = 3 m = 3×10^3 mm ; Central load (W) = 10 kN = 10×10^3 N ; Moment of inertia (I) = 12×10^6 mm^4 and modulus of elasticity (E) = 200 GPa = 200×10^3 N/mm^2.

Maximum slope of the beam

We know that maximum slope of the beam,

$$i_A = \frac{Wl^2}{16EI} = \frac{(10 \times 10^3) \times (3 \times 10^3)^2}{16 \times (200 \times 10^3) \times (12 \times 10^6)} = 0.0023 \text{ rad} \qquad \textbf{Ans.}$$

Maximum deflection of the beam

We also know that maximum deflection of the beam,

$$y_C = \frac{Wl^3}{48EI} = \frac{(10 \times 10^3) \times (3 \times 10^3)^3}{48 \times (200 \times 10^3) \times (12 \times 10^6)} = 2.3 \text{ mm} \qquad \textbf{Ans.}$$

A wooden beam 140 mm wide and 240 mm deep has a span of 4 m. Determine the load, that can be placed at its centre to cause the beam a deflection of 10 mm. Take E as 6 GPa.

SOLUTION. Given: Width (b) = 140 mm ; Depth (d) = 240 mm ; Span (l) = 4 m = 4×10^3 mm ; Central deflection (y_C) = 10 mm and modulus of elasticity (E) = 6 GPa = 6×10^3 N/mm^2.

Let W = Magnitude of the load,

We know that moment of inertia of the beam section,

$$I = \frac{bd^3}{12} = \frac{140 \times (240)^3}{12} = 161.3 \times 10^6 \text{ mm}^4$$

and deflection of the beam at its centre (y_C),

$$10 = \frac{Wl^3}{48EI} = \frac{W \times (4 \times 10^3)^3}{48 \times (6 \times 10^3) \times (161.3 \times 10^6)}$$

$$\therefore \quad W = \frac{10}{1.38 \times 10^{-3}} = 7.25 \times 10^3 \text{ N} = \textbf{7.25 kN} \quad \textbf{Ans.}$$

Simply Supported Beam with a Uniformly Distributed Load

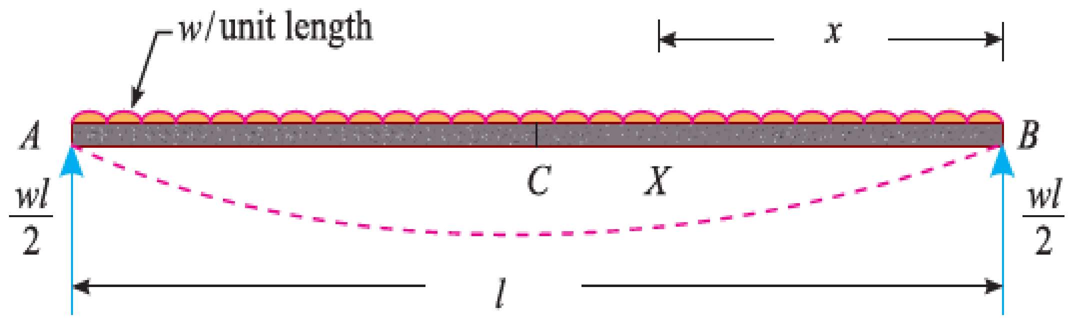


Fig. 19.6. Uniformly distributed load.

Consider a simply supported beam of length and carrying a uniformly distributed load of per unit length as shown in Fig. 19.6. From the geometry of the figure, we know that the reaction at *A*,

$$R_A = R_B = \frac{wl}{2}$$

Consider a section *X* at a distance *x* from *B*. We know that the bending moment at this section,

$$M_X = \frac{wlx}{2} - \frac{wx^2}{2} \qquad \text{...(Plus sign due to sagging)}$$

$$\therefore \qquad EI\frac{d^2y}{dx^2} = \frac{wlx}{2} - \frac{wx^2}{2} \qquad \text{...}(i)$$

Integrating the above equation,

$$EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^2}{6} + C_1 \quad \text{...(ii)}$$

where C_1 is the first constant of integration. We know when $x = \frac{l}{2}$, then $\frac{dy}{dx} = 0$

Substituting these values in the above equation,

$$0 = \frac{wl}{4}\left(\frac{l}{2}\right)^2 - \frac{w}{6}\left(\frac{l}{2}\right)^3 + C_1 = \frac{wl^3}{16} - \frac{wl^3}{48} + C_1$$

or $$C_1 = -\frac{wl^3}{24}$$

Substituting this value of C_1 in equation (*ii*),

$$\therefore \quad EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} - \frac{wl^3}{24} + C_1 \quad \text{...(iii)}$$

This is the required equation for the slope at any section. We know that maximum slope occurs at *A* and *B*. Thus for maximum slope, substituting $x = 0$ in equation (*iii*),

$$EI \cdot i_B = -\frac{wl^3}{24}$$

... (Minus sign means that the tangent at *A* makes an angle with *AB* in the negative or anticlockwise direction)

$$\therefore \quad i_B = -\frac{wl^3}{24EI}$$

or $$i_B = \frac{wl^3}{24EI}$$

By symmetry, $$i_A = \frac{wl^3}{24EI}$$

Integrating the equation (*iii*) once again,

$$EI \cdot y = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3x}{24} + C_2 \qquad \text{...(iv)}$$

where C_2 is the second constant of integration. We know when $x = 0$, then $y = 0$. Substituting these values in equation (*iv*), we get $C_2 = 0$

$$\therefore \qquad EI \cdot y = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3x}{24} \qquad \text{...(v)}$$

This is the required equation for the deflection at any section. We know that maximum deflection occurs at the mid-point C. Thus maximum deflection, substituting $x = l/2$ in equation (*v*),

$$EI \cdot y_C = \frac{wl}{12}\left(\frac{l}{2}\right)^3 - \frac{w}{24}\left(\frac{l}{2}\right)^4 - \frac{wl^3}{24}\left(\frac{l}{2}\right) = \frac{wl^4}{96} - \frac{wl^4}{384} - \frac{wl^4}{48} = -\frac{5wl^4}{384}$$

or $$y_C = -\frac{5wl^4}{384EI}$$ ...(Minus sign means that the deflection is downwards)

$$= \frac{5wl^4}{384EI}$$

A simply supported beam of span 4 m is carrying a uniformly distributed load of 2 kN/m over the entire span. Find the maximum slope and deflection of the beam. Take EI for the beam as 80 × 10^9 N-mm^2.

SOLUTION. Given: Span (l) = 4 m = 4 × 10^3 mm ; Uniformly distributed load (w) = 2 kN/m = 2 N/mm and flexural rigidity (E) = 80 × 10^9 N-mm^2.

Maximum slope of the beam

We know that maximum slope of the beam,

$$i_A = \frac{wl^3}{24EI} = \frac{2 \times (4 \times 10^3)^3}{34 \times (80 \times 10^9)} = \mathbf{0.067\ rad} \qquad \mathbf{Ans.}$$

Maximum deflection of the beam

We also know that maximum deflection of the beam,

$$y_C = \frac{5wl^4}{384EI} = \frac{5 \times 2 \times (4 \times 10^3)^4}{384 \times (80 \times 10^9)} = \mathbf{83.3\ mm} \qquad \mathbf{Ans.}$$

A simply supported beam of span 6 m is subjected to a uniformly distributed load over the entire span. If the deflection at the centre of the beam is not to exceed 4 mm, find the value of the load. Take E = 200 GPa and I = 300 × 10^6 mm^4.

SOLUTION. Given: Span (l) = 6 m = 6 × 10^3 mm ; Deflection at the centre (y_C) = 4 mm ; modulus of elasticity (E) = 200 GPa = 200 × 10^3 N/mm^2 and moment of inertia (I) = 300 × 10^6 mm^4.

Let w = Value of uniformly distributed load in N/mm or kN/m.

We know that deflection at the centre of the beam (y_C),

$$4 = \frac{5wl^4}{384EI} = \frac{5 \times w \times (6 \times 10^3)^4}{384 \times (200 \times 10^3) \times (300 \times 10^6)} = 0.281\,w$$

$$\therefore \quad w = \frac{4}{0.281} = 14.2 \text{ kN/m} \qquad \textbf{Ans.}$$

A timber beam of rectangular section has a span of 4.8 metres and is simply supported at its ends. It is required to carry a total load of 45 kN uniformly distributed over the whole span. Find the values of the breadth (b) and depth (d) of the beam, if maximum bending stress is not to exceed 7 MPa and maximum deflection is limited to 9.5 mm. Take E for timber as 10.5 GPa.

SOLUTION. Given: Span $(l) = 4.8 \text{ m} = 4.8 \times 10^3$ mm ; Total load $(W) = (wl) = 45 \text{ kN} = 45 \times 10^3$ N; Maximum bending stress $\sigma_{b\,(max)} = 7 \text{ MPa} = 7 \text{ N/mm}^2$; Maximum deflection $(y_C) = 9.5$ mm and modulus of elasticity $(E) = 10.5 \text{ GPa} = 10.5 \times 10^3 \text{ N/mm}^2$.

Let b = Breadth of the beam and

d = Depth of the beam.

We know that in a simply supported beam, carrying a uniformly distributed load, the maximum bending moment,

$$M = \frac{wl^2}{8} = \frac{wl \times l}{8} = \frac{W \times l}{8} = \frac{45 \times 4.8}{8}$$

$$= 27 \text{ kN-m} = 27 \times 10^6 \text{ N-mm}$$

and moment of inertia of a rectangular section,

$$I = \frac{bd^3}{12}$$

We also know that distance between the neutral axis of the section and extreme fibre,

$$y = \frac{d}{2}$$

∴ Maximum bending stress [$\sigma_{b\,(max)}$],

$$7 = \frac{M}{I} \times y = \frac{27 \times 10^6}{\frac{bd^3}{12}} \times \frac{d}{2} = \frac{162 \times 10^6}{bd^2}$$

or

$$bd^2 = \frac{162 \times 10^6}{7} = 23.14 \times 10^6$$

We know that maximum deflection (y_C),

$$9.5 = \frac{5wl^4}{384EI} = \frac{5\,(wl)\,l^3}{384EI} = \frac{5\,(45 \times 10^3) \times (4.8 \times 10^3)^3}{384 \times (10.5 \times 10^3) \times \frac{bd^3}{12}} = \frac{74.1 \times 10^9}{bd^3}$$

$$\therefore \quad bd^3 = \frac{74.1 \times 10^9}{9.5} = 7.8 \times 10^9$$

Dividing equation (*ii*) by equation (*i*),

$$d = \frac{7.8 \times 10^9}{23.14 \times 10^6} = \mathbf{337\ mm} \quad \textbf{Ans.}$$

Substituting this value of d in equation (*i*),

$$b \times (337)^2 = 23.14 \times 106$$

$$\therefore \quad b = \frac{23.14 \times 10^6}{(337)^2} = \mathbf{204\ mm} \quad \textbf{Ans.}$$

Cantilever with a Point Load at its Free End

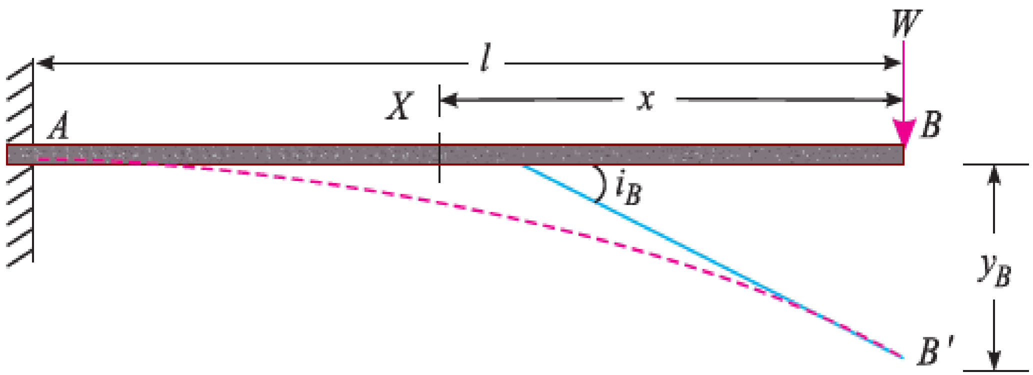


Fig. 20.1. Point load at the free end.

Consider a cantilever AB of length l and carrying a point load W at the free end as shown in Fig. 20.1. Consider a section X, at a distance x from the free end B.

We know that bending moment at this section,

$$M_X = -W \cdot x \qquad \text{...(Minus sign due to hogging)}$$

$$\therefore \quad EI\frac{d^2y}{dx^2} = -W \cdot x \qquad \text{...}(i)$$

Integrating the above equation,

$$EI\frac{dy}{dx} = -\frac{Wx^2}{2} + C_1 \qquad \text{...}(ii)$$

where C_1 is the first constant of integration. We know that when $x = l$, $\frac{dy}{dx} = 0$. Substituting these values in the above equation,

$$0 = -\frac{Wl^2}{2} + C_1 \quad \text{or} \quad C_1 = -\frac{Wl^2}{2}$$

Now substituting this value of C_1 in equation (*ii*),

$$EI\frac{dy}{dx} = -\frac{Wx^2}{2} + \frac{Wl^2}{2} \quad \text{...(iii)}$$

This is the required equation for the slope, at any section by which we can get the slope at any point on the cantilever. We know that maximum slope occurs at the free end. Now let us see the abbreviation i for the angle of inclination (in radian) and considering $i = \tan i$, for very small angles. Thus for maximum slope, substituting $x = 0$ in equation (*iii*),

$$EI \cdot i_B = \frac{Wl^2}{2}$$

$$\therefore \quad I_B = \frac{Wl^2}{2EI} \text{ radians}$$

Plus sign means that the tangent at B makes an angle with AB in the positive or clockwise direction. Integrating the equation (*iii*) once again,

$$EI \cdot y = \frac{Wx^3}{6} + \frac{Wl^2x}{2} + C_2 \quad \text{...(iv)}$$

bending

where C_2 is the second constant of integration. We know that when $x = l$, $y = 0$. Substituting these values in the above equation,

$$0 = -\frac{Wl^3}{6} + \frac{Wl^3}{2} + C_2 = \frac{Wl^3}{3} + C_2$$

or

$$C_2 = -\frac{Wl^3}{3}$$

...(Minus sign means that the deflection is downwards)

Substituting this value of C_2 in equation (*iv*),

$$EI \cdot y = -\frac{Wx^3}{6} + \frac{Wl^2x}{2} - \frac{Wl^3}{3}$$

$$= \frac{Wl^2x}{2} - \frac{Wx^3}{6} = \frac{Wl^3}{3} \quad \text{...(}v\text{)}$$

This is the required equation for the deflection, at any section. We know that maximum deflection occurs at the free end. Therefore for maximum deflection, substituting $x = 0$ in equation (*vi*),

$$EI \cdot y_B = -\frac{Wl^3}{3}$$

or

$$y_B = -\frac{Wl^3}{3EI}$$

$$= \frac{Wl^3}{3EI}$$

A cantilever beam 120 mm wide and 150 mm deep is 1.8 m long. Determine the slope and deflection at the free end of the beam, when it carries a point load of 20 kN at its free end. Take E for the cantilever beam as 200 GPa.

SOLUTION. Given: Width (b) = 120 mm; Depth (d) = 150 mm ; Span (l) = 1.8 m = 1.8×10^3 mm ; Point load (W) = 20 kN = 20×10^3 N and modulus of elasticity (E) = 200 GPa = 200×10^3 N/mm^2.

Slope at the free end

We know that moment of inertia of the beam section,

$$I = \frac{bd^3}{12} = \frac{120 \times (150)^3}{12} = 33.75 \times 10^6 \text{ mm}^4$$

and slope at the free end,

$$i_B = \frac{Wl^2}{2EI} = \frac{(20 \times 10^3) \times (1.8 \times 10^3)^2}{2 \times (200 \times 10^3) \times (33.75 \times 10^6)} = \mathbf{0.0048 \text{ rad}} \quad \textbf{Ans.}$$

Deflection at the free end

We also know that deflection at the free end,

$$y_B = \frac{Wl^3}{3EI} = \frac{(20 \times 10^3) \times (1.8 \times 10^3)^3}{3 \times (200 \times 10^3) \times (33.75 \times 10^6)} = \mathbf{5.76 \text{ mm}} \quad \textbf{Ans.}$$

Macaulay's Method* for Slope and Deflection

We have seen in the previous articles and examples that the problems of deflections in beams are bit tedious and laburious, specially when the beam is carrying some point loads. Mr. W.H. Macaulay devised a method, a continuous expression, for bending moment and integrating it in such a way, that the constants of integration are valid for all sections of the beam ; even though the law of bending moment varies from section to section. Now we shall discuss the application of Macaulay's method for finding out the slopes and deflection of a few types of beams:

NOTES. The following rules are observed while using Macaulay's method:

1. Always take origin on the extreme left of the beam.
2. Take left clockwise moment as negative and left anticlockwise moment as positive.
3. While calculating the slopes and deflections, it is convenient to use the values first in terms of kN and metres.

Simply supported beam with a central point load.

Consider a simply supported beam AB of length l and carrying a point load W at the centre of the beam C as shown in Fig. 19.8.

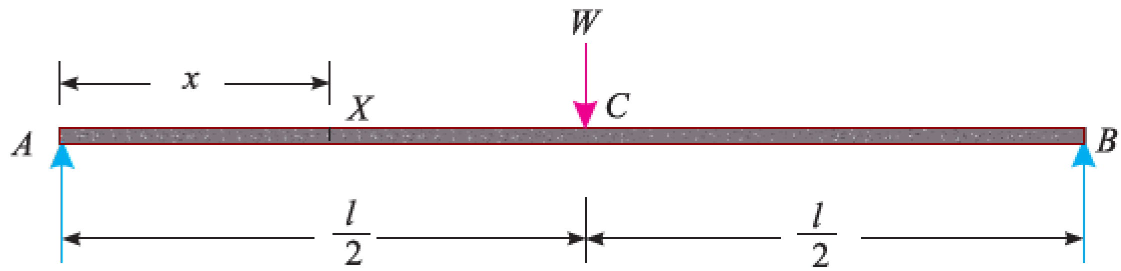


Fig. 19.8

Take A as the origin. We know that bending moment at any point, in section AC at a distance x from A,

$$M_X = -\frac{W}{2}x \quad \text{....(Minus sign due to left clockwise)}$$

and the bending moment at any point in section CB and at a distance x from A,

$$M_X = -\frac{W}{2}x + W\left(x - \frac{1}{2}\right) \quad ...(i)$$

Thus we can express the bending moment, for all the sections of the beam in a single equation, *i.e.*,

$$M_x = -\frac{W}{2}x \;\vdots\; + W\left(x - \frac{1}{2}\right)$$

For any point in section *AC*, stop at the dotted line, and for any point in section *CB* add the expression beyond the dotted line also.

Now re-writing the above equation,

$$EI\,\frac{d^2y}{dx^2} = -\frac{Wx}{2} \;\vdots\; + W\left(x - \frac{1}{2}\right) \qquad \text{...}(ii)$$

Integrating the above equation,

$$EI\,\frac{dy}{dx} = -\frac{Wx^2}{4} + C_1 \;\vdots\; + \frac{W}{2}\left(x - \frac{1}{2}\right) \qquad \text{...}(iii)$$

It may be noted that the integration of $\left(x - \frac{l}{2}\right)$ has been made as a whole and not for individual terms for the expression. This is only due to this simple integration that the Macaulay's method is more effective. This type of integration is also justified as the constant of integration C_1 is not only valid for the section *AC*, but also for section *CB*.

Integrating the equation (*iii*) once again,

$$EI \cdot y = -\frac{Wx^3}{12} + C_1\,x + C_2 \;\vdots\; + \frac{W}{6}\left(x - \frac{l}{2}\right)^3 \qquad \text{...}(iv)$$

It may again be noted that the integration of $\left(x-\frac{l}{2}\right)^2$ has again been made as a whole and not for individual terms. We know that when $x = 0$, then $y = 0$. Substituting these values in equation (*iv*), we find $C_2 = 0$. We also know that when $x = l$, then $y = 0$. Substituting these values of x and y and $C_2 = 0$ in equation (*iv*),

$$0 = -\frac{Wl^3}{12} + C_1 l + \frac{W}{6}\left(\frac{l}{2}\right)^3$$

$$\therefore \qquad C_1\, l = -\frac{Wl^3}{12} - \frac{Wl^3}{48} = \frac{3Wl^3}{48} = \frac{Wl^3}{16}$$

or

$$C_1 = \frac{Wl^2}{16}$$

Now substituting this value of C_1 in equation (*iii*),

$$\therefore \qquad EI\frac{dy}{dx} = \frac{Wx^2}{4} + \frac{Wl^2}{16} \;\vdots\; + \frac{W}{2}\left(x-\frac{l}{2}\right)^2$$

This is the required equation for slope at any section. We know that maximum slope occurs at A and B. Thus for maximum slope at A, substituting $x = 0$ in equation (*v*) upto the dotted line only,

$$EI \cdot i_A = \frac{WL^2}{16}$$

By symmetry, $$i_B = \frac{Wl^2}{16EI} \quad \text{...(As before)}$$

Substituting the value of C_1 again in equations (*iv*) and $C_2 = 0$,

$$EI \cdot y = -\frac{Wx^3}{12} + \frac{Wl^2 x}{16} \;\vdots\; + \frac{W}{6} + \left(x - \frac{l}{2}\right)^3 \quad \text{...(vi)}$$

This is required equation for deflection at any section. We know that maximum deflection occurs at *C*. Thus for maximum deflection, substituting $x = l/2$ in equation (*vi*) for the portion *AC* only (remembering that *C* lies in *AC*),

$$EI \cdot y_C = -\frac{W}{12}\left(\frac{l}{2}\right)^3 \;\vdots\; + \frac{Wl^2}{16}\left(\frac{l}{2}\right) = \frac{Wl^3}{48}$$

or $$y_C = \frac{Wl^3}{48EI} \quad \text{...(As before)}$$

A horizontal steel girder having uniform cross-section is 14 m long and is simply supported at its ends. It carries two concentrated loads as shown in Fig. 19.10.

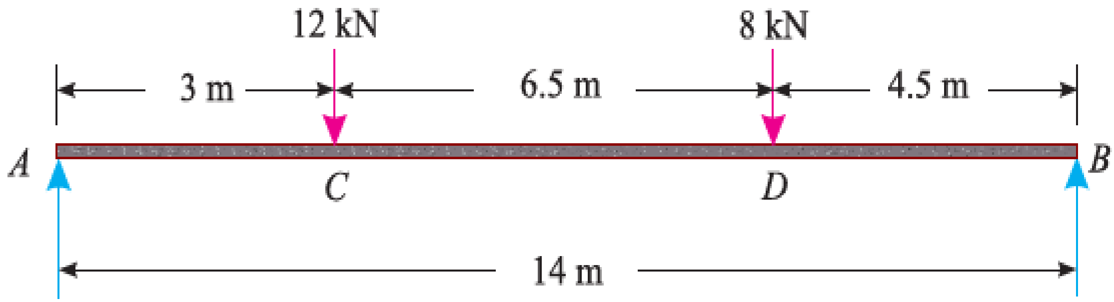


Fig. 19.10

Calculate the deflections of the beam under the loads C and D. Take E = 200 GPa and I = 160 × 10⁶ mm⁴.

SOLUTION. Given: Span (l) = 14 m = 14×10^3 mm ; Load at C (W_1) = 12 kN = 12×10^3 N ; Load at D (W_2) = 8 kN = 8×10^3 N ; Modulus of elasticity (E) = 200 GPa = 200×10^3 N/mm² and moment of inertia (I) = 160×10^6 mm⁴.

Taking moments about A and equating the same,

$$R_B \times 14 = (12 \times 3) + (8 \times 9.5) = 112$$

$$\therefore \quad R_B = \frac{112}{14} = 8 \text{ kN} = 8 \times 10^3 \text{ N}$$

and

$$R_A = (12 + 8) - 8 = 12 \text{ kN} = 12 \times 10^3 \text{ N}$$

Now taking A as the origin and using Macaulay's method, the bending moment at any section X at a distance x from A,

$$EI\frac{d^2y}{dx^2} = -(12\times10^3)\,x + \;\vdots\; (12\times10^3)\times[x-(3\times10^3)]$$

$$+ (8\times10^3)\times[x-(9.5\times10^3)]$$

Integrating the above equation,

$$EI\frac{dy}{dx} = -(12\,t\,10^3)\frac{x^2}{2} + \;\vdots\; C_1 + (12\times10^3)\times\frac{[x-(3\times10^3)]^2}{2}$$

$$+ (8\times10^3)\times\frac{[x-(9.5\times10^3)]^2}{2}$$

$$= -(6\times10^3)\,x^2 + C_1 \;\vdots\; + (6\times10^3)\times[x-(3\times10^3)]^2$$

$$\vdots\; + (4\times10^3)\times[x-(9.5\times10^3)] \qquad \ldots(i)$$

Integrating the above equation once again,

$$EI\cdot y = -(6\times10^3)\times\frac{x^3}{3} + C_1\,x + C_2 + \;\vdots\; (6\times10^3)\times\frac{[x-(3\times10^3)]^3}{3}$$

$$\vdots\; + (4\times10^3)\times\frac{[x-(9.5\times10^3)]^3}{3}$$

$$= (2\times10^3)\,x^3 + C_1\,x + C^2 \;\vdots\; + (2\times10^3)\,[x-(3\times10^3)]^3$$

$$+\frac{4\times10^3}{3}\times(x-(9.5\times10^3)]^3 \qquad ...(ii)$$

We know that when $x = 0$, then $y = 0$. Therefore $C_2 = 0$. And when $x = (14 \times 10^3)$ mm, then $y = 0$. Therefore

$$0 = -(2\times10^3)\times(14\times10^3)^3 + C_1\times(14\times10^3)$$
$$+ (2\times10^3)\times[(14\times10^3)-(3\times10^3)]^3$$
$$+\frac{4\times10^3}{3}\times[(14\times10^3)\ \ (9.5\times10^3)]^3$$
$$= -(5488\times10^{12}) + (14\times10^3)\,C_1 + (2662\times10^{12}) + 121.5\times10^{12}$$
$$= -(2704.5\times10^{12}) + (14\times10^3)\,C_1$$

$$\therefore \qquad C_1 = \frac{2704.5\times10^{12}}{14\times10^3} = 193.2\times10^9$$

Substituting the value of C_1 equal to 193.2×10^9 and $C_2 = 0$ in equation (*ii*),

$$EIy = -2\times10^3\,x^3 + 193.2\times10^9\,x \quad \vdots \quad + 2\times10^3\,[x-(3\times10^3)]3$$
$$\vdots\ +\frac{4\times10^3}{3}\times[x-(9.5\times10^3)]^3 \qquad ...(iii)$$

Now for deflection under the 12 kN load, substituting $x = 3$ m (or 3×10^3 mm) in equation (*iii*) up to the first dotted line only,

$$EIy_C = -2\times10^3\times(3\times10^3)^3 + 193.2\times10^9\times(3\times10^3)$$
$$= -(54\times10^{12}) + (579.6\times10^{12}) = 525.6\times10^{12}$$

$$\therefore \qquad y_C = \frac{525.6\times10^{12}}{EI} = \frac{525.6\times10^{12}}{(200\times10^3)\times(160\times10^6)} = \mathbf{16.4\ mm} \qquad \textbf{Ans.}$$

Similarly, for deflection under the 8 kN load, substituting $x = 9.5$ m (or 9.5×10^3 mm) in equation (*iii*) up to the second dotted line only,

$$EI\,y_D = -2 \times 10^3 \times (9.5 \times 10^3)^3 + 193.2 \times 10^9 \times (9.5 \times 10^3) + 2 \times 10^3 \times [(9.5 \times 10^3) - (3 \times 10^3)]^3$$

$$= -(1714.75 \times 10^{12}) + (1835.4 \times 10^{12}) + (549.25 \times 10^{12})$$

$$= 669.9 \times 10^{12}$$

$$\therefore \quad y_D = \frac{669.9 \times 10^{12}}{EI} = \frac{669.9 \times 10^{12}}{(200 \times 10^3) \times (160 \times 10^6)} = \mathbf{20.9\ mm} \quad \textbf{Ans.}$$

A horizontal beam AB is freely supported at A and B, 8 m apart and carries a uniformly distributed load of 15 kN/m run (including its own weight). A clockwise moment of 160 kN-m is applied to the beam at a point C, 3 m from the left hand support A. Calculate the slope of the beam at C, if $EI = 40 \times 10^3$ kN-m^2.

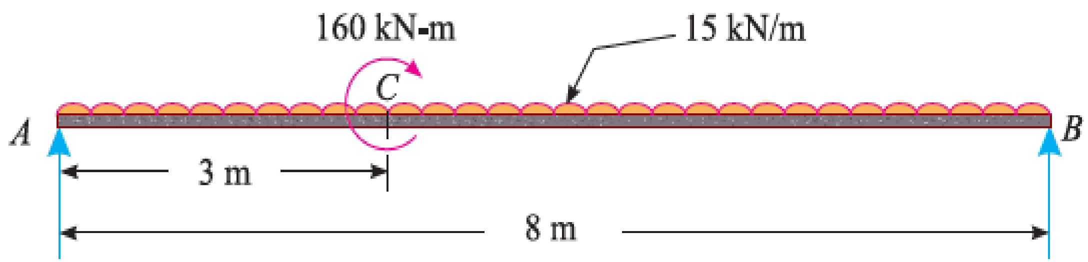


Fig. 19.11

Solution. Given: Span (l) = 8 m ; Uniformly distributed load (w) = 15 kN/m ; Moment at C (μ) = 160 kN-m (clockwise) and flexural rigidity (EI) = 40×10^3 kN-m^2.

Taking moments about A and equating the same,

$$R_B \times 8 = (15 \times 8 \times 4) + 160 = 640 \text{ kN-m}$$

$$\therefore \quad R_B = \frac{640}{8} = 80 \text{ kN}$$

and $$R_A = (15 \times 8) - 80 = 40 \text{ kN}$$

Let $$i_C = \text{Slope at } C.$$

Taking A as origin and using Macaulay's method, the bending moment at any section X at a distance x from A,

$$EI\frac{d^2y}{dx^2} = -40x \;\vdots\; +15x\times\frac{x}{2} - 160\,(x-3)$$

$$= -40x \;\vdots\; \frac{15x^2}{2} - 160\,(x-3)$$

Integrating the above equation,

$$EI\frac{dy}{dx} = -40\frac{x^2}{2} + C_1 \;\vdots\; +\frac{15x^3}{6} \;\vdots\; -160\,(x-3)$$

$$= -20x^2 + C_1 \;\vdots\; +\frac{5x^3}{2} \;\vdots\; -160\,(x-3) \qquad \text{...}(i)$$

Integrating the above equation once again,

$$EI\cdot y = -\frac{20x^3}{3} + C_1x + C_2 \;\vdots\; +\frac{5x^4}{8} - \frac{160\,(x-3)^2}{2} \qquad \text{...}(ii)$$

We know that when $x = 0$, then $y = 0$. Therefore $C_2 = 0$ and when $x = 8$, then $y = 0$. Therefore

$$0 = \frac{-20 \times (8)^3}{3} + (C_1 \times 8) + \frac{5 \times (8)^4}{8} - \frac{160 \times (5)^2}{2}$$

$$= 8C_1 - 2853.3$$

$$\therefore \quad C_1 = \frac{2853.3}{8} = 356.7$$

Substituting the values of $C_1 = 356.7$ and $C_2 = 0$ in equation (*i*),

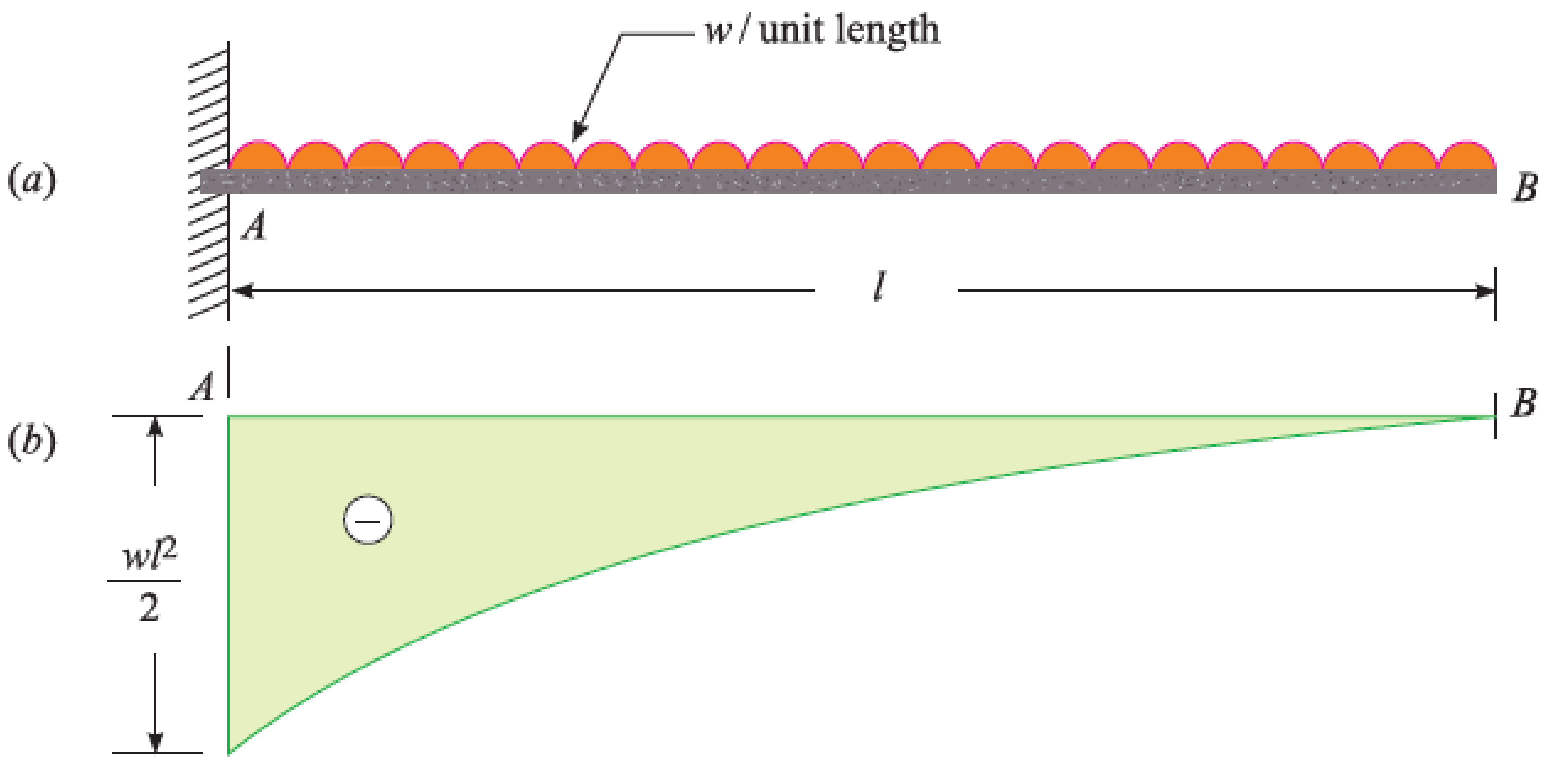
$$EI\,\frac{dy}{dx} = -20x^2 + 356.7 \;\vdots\; + \frac{5x^3}{2} \;\vdots\; -160\,(x-3)$$

Now for the slope at C, substituting $x = 3$ m in the above equation up to C *i.e.*, neglecting the *last term.

$$EI \cdot i_C = -20 \times 3^2 + 356.7 + \frac{5 \times 3^3}{2} = 244.2$$

$$\therefore \quad i_C = \frac{244.2}{40 \times 10^3} = \mathbf{0.0061\ rad} \qquad \textbf{Ans.}$$

Cantilever with a Uniformly Distributed Load



Consider a cantilever *AB* of length *l*, and carrying a uniformly distributed load of *w* per unit length as shown in Fig. 21.11 (*a*).

We know that the bending moment will be zero at and will increase in the form of a parabola to $\frac{wl^2}{2}$ at *A* as shown in Fig. 21.11 (*b*). Therefore area of bending moment diagram,

$$A = \frac{wl^2}{2} \times l \times \frac{1}{3} = \frac{Wl^3}{6}$$

and distance between the centre of gravity of bending moment diagram and *B*,

$$\bar{x} = \frac{3l}{4}$$

$$\therefore \quad i_B = \frac{A}{EI} = \frac{wl^3}{6EI} \text{ radians} \quad \text{...(As before)}$$

and

$$y_B = \frac{A\bar{x}}{EI} = \frac{\frac{wl^3}{6} \times \frac{3l}{4}}{EI} = \frac{wl^4}{8EI} \quad \text{...(As before)}$$

A cantilever beam 120 mm wide and 150 mm deep carries a uniformly distributed load of 10 kN/m over its entire length of 2.4 meters. Find the slope and deflection of the beam at its free end. Take E = 180 GPa.

SOLUTION. Given : Width (b) = 120 mm ; Depth (d) = 150 mm ; Uniformly distributed load (w) = 10 kN/m = 10 N/mm ; Length (l) = 2.4 m = 2.4×10^3 mm and modulus of elasticity (E) = 180 GPa = 180×10^3 N-mm^2.

Slope at the free end of the beam

We know that moment of inertia of the cantilever beam section,

$$I = \frac{bd^3}{12} = \frac{120 \times (150)^3}{12} = 33.75 \times 10^6 \text{ mm}^4$$

and slope at the free end,

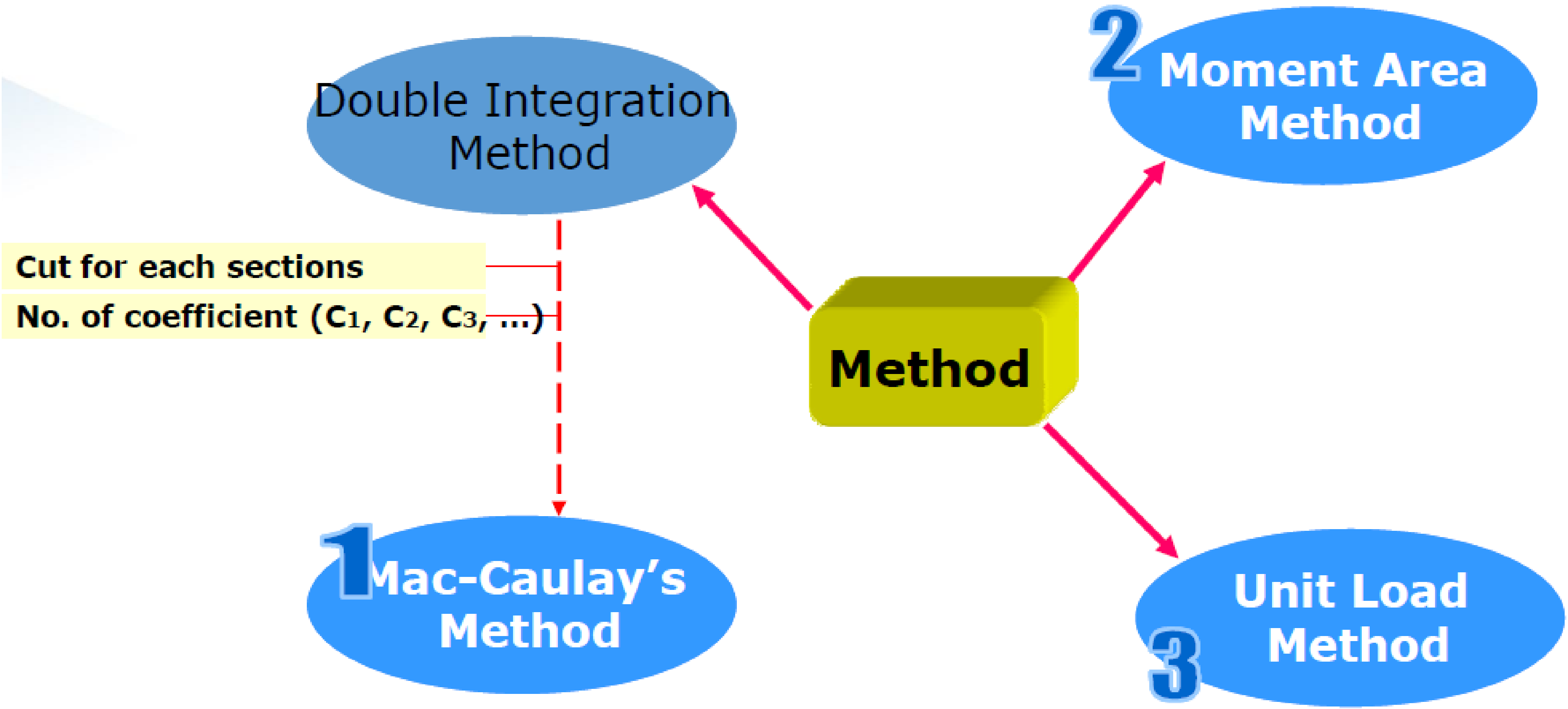
$$i_B = \frac{wl^3}{6\,EI} = \frac{10 \times (2.4 \times 10^3)^3}{6 \times (180 \times 10^3) \times (33.75 \times 10^6)} = 0.0038 \text{ rad} \qquad \textbf{Ans.}$$

Deflection at the free end of the beam

We also know that deflection at the free end,

$$y_B = \frac{wl^4}{8\,EI} = \frac{10 \times (2.4 \times 10^3)^4}{8 \times (180 \times 10^3) \times 33.75 \times 10^6)} = 6.83 \text{ mm} \qquad \textbf{Ans.}$$

Basic methods to find deflection for statically determinate beams:



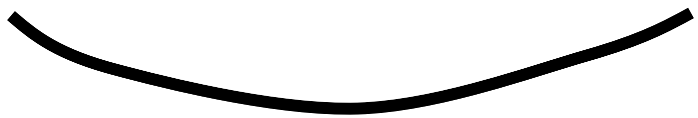
Double Integration Method

A technique used in structural analysis to determine the deflection of Euler-Bernoulli beams

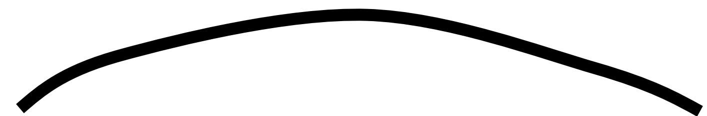
- The first English language description of the method was by Macaulay.
- The actual approach appears to have been developed by Clebsch in 1862
- The double integration method is a powerful tool in solving deflection and slope of a beam at any point because we will be able to directly work on the equation of the elastic curve.

Double Integration Method

Sign Convention



Positive Bending



Negative Bending

Assumptions and Limitations

- Deflections caused by shearing action negligibly small compared to bending
- Deflections are small compared to the cross-sectional dimensions of the beam
- All portions of the beam are acting in the elastic range
- Beam is straight prior to the application of loads

Double Integration Method

The governing differential equation is defined as

$$M = EI\frac{d^2y}{dx^2} \quad \text{or} \quad \frac{M}{EI} = \frac{d^2y}{dx^2}$$

on integrating one get,

$$\frac{dy}{dx} = \int \frac{M}{EI}dx + A$$ - - - - this equation gives the slope of the loaded beam.

Integrate once again to get the deflection.

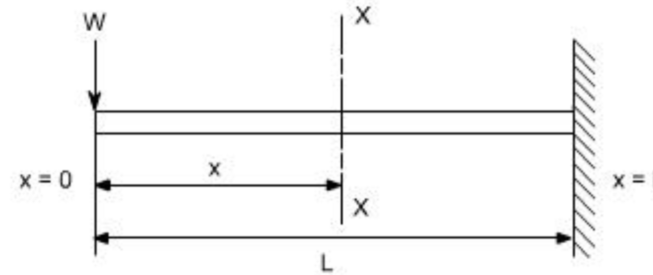
$$y = \int\int \frac{M}{EI}dx + Ax + B$$

Where A and B are constants of integration to be evaluated from the known conditions of slope and deflections for the particular value of x.

Example 1

Case 1: Cantilever Beam with Concentrated Load at the end:-

A cantilever beam is subjected to a concentrated load W at the free end, it is required to determine the deflection of the beam



Consider any X-section X-X located at a distance x from the left end or the reference

The expressions for the shear force abd the bending moment

$S.F|_{x-x} = -W$

$B.M|_{x-x} = -W.x$

Therefore $M|_{x-x} = -W.x$

the governing equation $\frac{M}{EI} = \frac{d^2y}{dx^2}$

substituting the value of M interms of x then integrating the equation one get

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = -\frac{Wx}{EI}$$

$$\int \frac{d^2y}{dx^2} = \int -\frac{Wx}{EI}dx$$

$$\frac{dy}{dx} = -\frac{Wx^2}{2EI} + A$$

Integrating once more,

$$\int \frac{dy}{dx} = \int -\frac{Wx^2}{2EI}dx + \int A\,dx$$

$$y = -\frac{Wx^3}{6EI} + Ax + B$$

Example 1: Case 1

The constants A and B are required to be found out by utilizing the boundary conditions as defined below

i.e at x= L ; y= 0 -------------------- (1)

at x = L ; dy/dx = 0 -------------------- (2)

Utilizing the second condition, the value of constant A is obtained as

$$A = \frac{Wl^2}{2EI}$$

While employing the first condition yields

$$y = -\frac{WL^3}{6EI} + AL + B$$

$$B = \frac{WL^3}{6EI} - AL$$

$$= \frac{WL^3}{6EI} - \frac{WL^3}{2EI}$$

$$= \frac{WL^3 - 3WL^3}{6EI} = -\frac{2WL^3}{6EI}$$

$$B = -\frac{WL^3}{3EI}$$

Substituting the values of A and B we get

$$y = \frac{1}{EI}\left[-\frac{Wx^3}{6EI} + \frac{WL^2x}{2EI} - \frac{WL^3}{3EI}\right]$$

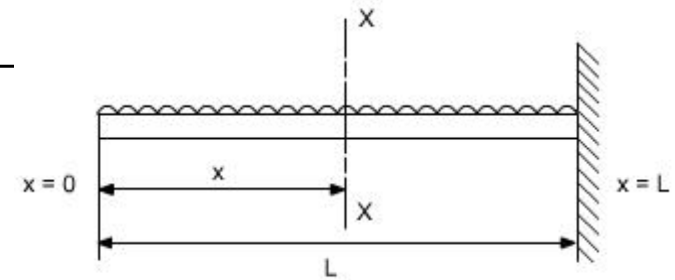
The slope as well as the deflection would be maximum at the free end hence putting x=0 we get,

$$\boxed{y_{max} = -\frac{WL^3}{3EI}}$$

$$\boxed{(\text{Slope})_{maxm} = +\frac{WL^2}{2EI}}$$

Example 1: Case 2

Case 2: A Cantilever with Uniformly distributed



In this case the cantilever beam is subjected to U.d.l with rate of intensity varying w/length. The same procedure can also be adopted in this case

$$S.F|_{x-x} = -w$$

$$B.M|_{x-x} = -w.x.\frac{x}{2} = w\left(\frac{x^2}{2}\right)$$

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = -\frac{wx^2}{2EI}$$

$$\int \frac{d^2y}{dx^2} = \int -\frac{wx^2}{2EI}\,dx$$

$$\frac{dy}{dx} = -\frac{wx^3}{6EI} + A$$

$$\int \frac{dy}{dx} = \int -\frac{wx^3}{6EI}\,dx + \int A\,dx$$

$$y = -\frac{wx^4}{24EI} + Ax + B$$

Boundary conditions relevant to the problem are as follows:

1. At x = L; y = 0
2. At x= L; dy/dx = 0

The second boundary conditions yields

$$A = +\frac{wx^3}{6EI}$$

whereas the first boundary conditions yields

$$B = \frac{wL^4}{24EI} - \frac{wL^4}{6EI}$$

$$B = -\frac{wL^4}{8EI}$$

$$\text{Thus, } y = \frac{1}{EI}\left[-\frac{wx^4}{24} + \frac{wL^3x}{6} - \frac{wL^4}{8}\right]$$

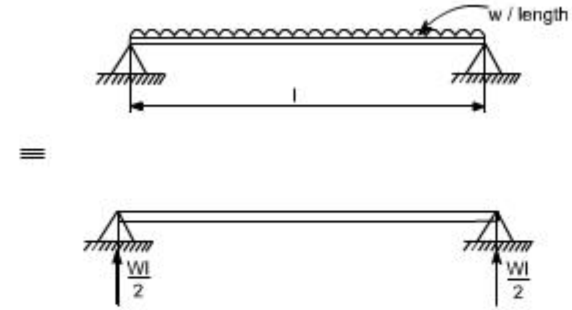
So y_{max^m} will be at $x = 0$

$$\boxed{y_{max^m} = -\frac{wL^4}{8EI}}$$

$$\boxed{\left(\frac{dy}{dx}\right)_{max^m} = \frac{wL^3}{6EI}}$$

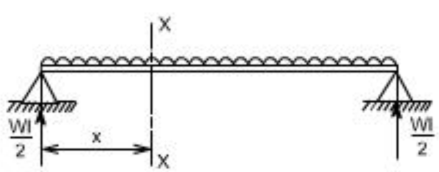
Example 1: Case 3

Case 3: Simply Supported beam with uniformly distributed Loads



In this case a simply supported beam is subjected to a uniformly distributed load whose rate of intensity varies as w/length.

In order to write down the expression for bending moment consider any cross-section at distance of x metre from left end support



$$S.F|_{x-x} = w\left(\frac{l}{2}\right) - w.x$$

$$B.M|_{x-x} = w.\left(\frac{l}{2}\right).x - w.x.\left(\frac{x}{2}\right)$$

$$= \frac{wl.x}{2} - \frac{wx^2}{2}$$

The differential equation which gives the elastic curve for the deflected beam is

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = \frac{1}{EI}\left[\frac{wl.x}{2} - \frac{wx^2}{2}\right]$$

$$\frac{dy}{dx} = \int \frac{wlx}{2EI}dx - \int \frac{wx^2}{2EI}dx + A$$

$$= \frac{wlx^2}{4EI} - \frac{wx^3}{6EI} + A$$

Integrating, once more one gets

$$y = \frac{wlx^3}{12EI} - \frac{wx^4}{24EI} + A.x + B \qquad \text{-----(1)}$$

Example 1: Case 3

Boundary conditions which are relevant in this case are that the deflection at each support must be zero.

i.e. at x = 0; y = 0 : at x = l; y = 0

Let us apply these two boundary conditions on equation (1) because the boundary conditions are on y, This yields B = 0.

$$0 = \frac{wl^4}{12EI} - \frac{wl^4}{24EI} + A.l$$

$$A = -\frac{wl^3}{24EI}$$

So the equation which gives the deflection curve is

$$y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

In this case the maximum deflection will occur at the centre of the beam where x = L/2 [i.e. at the position where the load is being applied].So if we substitute the value of x = L/2

Then $$y_{max^m} = \frac{1}{EI}\left[\frac{wL}{12}\left(\frac{L^3}{8}\right) - \frac{w}{24}\left(\frac{L^4}{16}\right) - \frac{wL^3}{24}\left(\frac{L}{2}\right)\right]$$

$$\boxed{y_{max^m} = -\frac{5wL^4}{384EI}}$$

Example 1: Case 3

Conclusions:

i. The value of the slope at the position where the deflection is maximum would be zero.

ii. The value of maximum deflection would be at the centre i.e. at x = L/2.

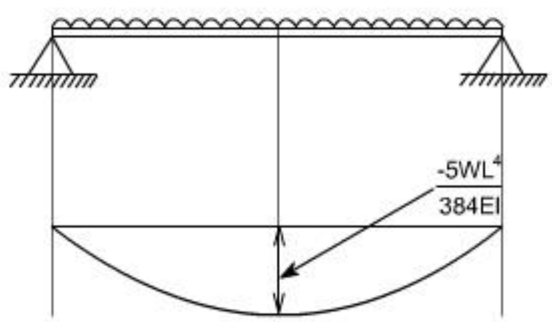
The final equation which is governs the deflection of the loaded beam in this case is

$$y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

By successive differentiation one can find the relations for slope, bending moment, shear force and rate of loading

Deflection (y)

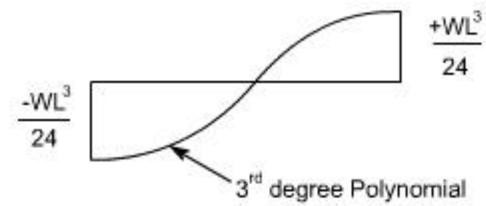
$$yEI = \left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$



Example 1: Case 3

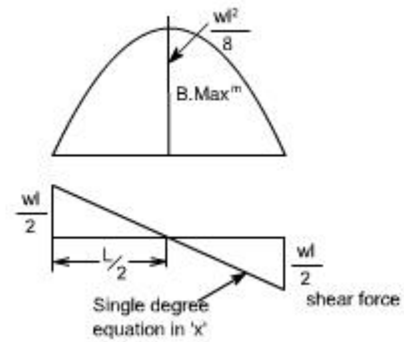
Slope (dy/dx)

$$EI.\frac{dy}{dx} = \left[\frac{3wLx^2}{12} - \frac{4wx^3}{24} - \frac{wL^3}{24}\right]$$



Bending Moment

$$\frac{d^2y}{dx^2} = \frac{1}{EI}\left[\frac{wLx}{2} - \frac{wx^2}{2}\right]$$



Shear force is obtained by taking third derivative.

$$EI\frac{d^3y}{dx^3} = \frac{wL}{2} - w.x$$

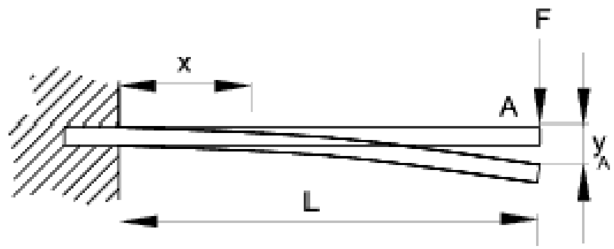
Rate of intensity of loading

$$EI\frac{d^4y}{dx^4} = -w$$

Example 2: Cantilever Beam

Example - Cantilever beam

Consider a cantilever beam (uniform section) with a single concentrated load at the end. At the fixed end x = 0, dy = 0 , dy/dx = 0



From the equilibrium balance ..At the support there is a resisting moment -FL and a vertical upward force F.
At any point x along the beam there is a moment $F(x - L) = M_x = EI\, d^2y/dx^2$

$$EI\frac{d^2y}{dx^2} = -F\,(L-x) \quad \text{Integrating}$$

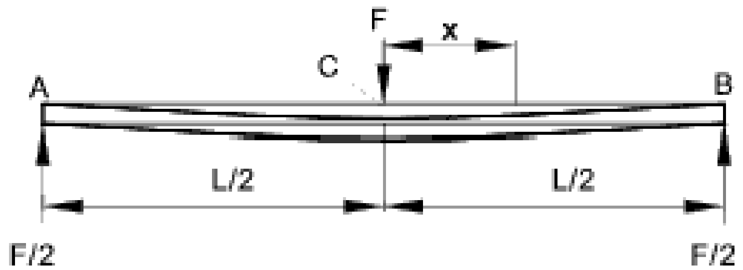
$$EI\frac{dy}{dx} = -F\,(Lx - \frac{x^2}{2}) + C_1 \quad \text{.....}(C_1\text{=0 because } dy/dx = 0 \text{ at } x = 0\,)$$

Integrating again

$$EI\,y = -F\,(\frac{Lx^2}{2} - \frac{x^3}{6}) + C_2 \quad \text{.....}(C\text{ =0 because } y = 0 \text{ at } x = 0\,)$$

$$\text{At end A} \quad \left(\frac{dy}{dx}\right)_A = -\frac{F}{EI}(L^2 - \frac{L^2}{2}) = -\frac{FL^2}{2EI} \quad \text{and} \quad y_A = -\frac{F}{EI}(\frac{L^3}{2} - \frac{L^3}{6}) = -\frac{FL^3}{3EI}$$

Example 2: Cantilever Beam



$$\frac{d^2y}{dx^2} = \frac{1}{EI}\left[\frac{F}{2}\left(\frac{L}{2}+x\right) - Fx\right] = \frac{F}{2EI}\left(\frac{L}{2}-x\right)$$ Integrating

$$\frac{dy}{dx} = \frac{F}{2EI}\left(\frac{Lx}{2} - \frac{x^2}{2}\right) + C_1 ..$$ ($C_1 = 0$ because $dy/dx = 0$ at $x = 0$)

Integrating again $y = \frac{F}{2EI}\left(\frac{Lx^2}{4} - \frac{x^3}{6}\right) + C_2$

$y = 0$ when $x = L/2$ therefore $\frac{F}{2EI}\left(\frac{L^3}{8} - \frac{L^3}{12}\right) + c_2 = 0$

and thus $c_2 = -\frac{FL^3}{48EI}$

At end B $\left(\frac{dy}{dx}\right)_b = -\frac{F}{2EI}\left(\frac{L^2}{4} - \frac{L^2}{8}\right) = \frac{FL^2}{16EI}$ and $y_B = \frac{F}{2EI}\left(\frac{L^3}{8} - \frac{L^3}{12}\right) - \frac{FL^3}{48EI} = 0$

At centre C $x = 0$ $y_c = -\frac{Fl^3}{48EI}$ (slope $\frac{dy}{dx} = 0$ by symmetry)

Mac-Caulay's Method

Mac-Caulay's method is a means to find the equation that describes the deflected shape of a beam

From this equation, any deflection of interest can be found

Mac-Caulay's method enables us to write a single equation for bending moment for the full length of the beam

When coupled with the Euler-Bernoulli theory, we can then integrate the expression for bending moment to find the equation for deflection using the double integration method.